Risky vs Safe production mode: when to invest and when to switch?

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Risky vs Safe production mode: when to invest and when to switch?

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Abstract

Moments of crisis, as the pandemic situation in 2020/2021 or the financial and economic crisis of 2007/2008, show that the survival of companies depends very much on the way they adjust to the market state, being able to adopt different operation modes. The question is then when to switch from one operating mode to another. Firms that are not yet in the market have to decide the optimal time to invest, and in which mode (risky or safe) they will invest, considering the future switching strategy.

In this paper we study both the switching and the investment problems using the Real Options framework. We characterise the optimal strategies in terms of the triggers, providing the corresponding value functions. We also study the influence of the costs and the parameters of the underlying stochastic process in the optimal strategies. Finally, we discuss how the firm can choose the safe mode in order to optimise its revenues.

Under some conditions, the firm can operate with a negative instantaneous profit. This region - the hysteresis region - can only be reached under special conditions. We present two situations where the firm may produce or even invest in this region.

Subject Classification: Decision analysis, Finance, Dynamic programming/Optimal control

Contextualising the subject classification:
Decision analysis: Producing in a risky/safe mode: what's the optimal strategy?
Finance: Investing in one of two operating modes: optimal strategy.
Dynamic programming/Optimal control: Optimal switching and stopping problems: an application.

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1 Introduction

In this paper we consider a firm that may operate in two modes, being able to switch between them as many times as needed, paying some pre-defined costs. Moreover, it has also the option to permanently leave the market, and this option is possible to exercise in both modes. The two projects differ in terms of the *risk* associated with the payoff: one being more profitable when the market conditions are favourable but leading to larger losses in times of crisis (risky mode), whereas the other mode leads to smaller profits and losses (safe mode). Therefore the firm may adjust itself, in terms of production mode, to the conditions of the market.

We start by deriving the optimal operational mode for a firm already active in the market, defining when the firm should switch from one mode to other, in order to maximise expected profits.

This problem - which we call the *switching problem* - becomes more relevant when firms have to face volatile markets or imminent social, economic or financial crisis. In these situations, firms have to adapt fast to the changing market conditions, adjusting, in particular, their production processes. In the 2008 crisis, many firms felt the need to adjust their production processes in order to face declining markets and to avoid large losses. One of the strategies followed by companies to decrease the costs associated with the production was the layoff, which in fact represents a change in the production mode, using our terminology.

During this crisis, many companies adopted this type of strategy, reducing the risk of having large losses. Firms like Merck, General Electric, Whirpool and others are examples of such decision. Of course as soon as the demand started increasing, firms needed to change their strategy, in order to accommodate larger demand and therefore to increase their profits.

Actually, this question has gained more importance, with the current situation of the pandemic of COVID-19 caused by the SARS-COV-2, as companies are scrambling to mobilise responses. Crisis like this one have a dynamic trajectory, which requires a constant reframing of models and plans. According to Reeves et al. (2020), the Corona crisis has shown that companies need to be very flexible, reallocating labour flexibly to different activities, and acting proactively. For instance, in China many companies decided to reallocate employees to new and valuable activities instead of considering the layoff strategy. Also, they deployed sales efforts to other channels, considering, for instance, online sales. After the peak of the crisis has passed, these companies need to adjust again to the new situation, preparing for a faster recovery.

In Europe, companies including Zara, Nivea and Dyson, made drastic changes in their production to help with the response to the COVID-19 pandemic, producing needed equipment's. Additionally, they keep trying to survive to the outbreak. Another example is the engineering firm Meggitt, which changed to produce thousands of ventilators to treat patients with COVID-19 who develop severe respiratory problems. Restaurants, pubs and coffees, closed since the outbreak of the pandemic, moved to services like takeaway and home-delivery (these examples can be found in Davies and O'Carroll (2020)).

These examples show companies that are already in the market and need to take decisions regarding their production mode. In this paper, we also consider the optimal investment strategy for firms that intend to undertake an investment opportunity. The optimal investment decision will define the timing and the production mode that the firm will choose for its investment. After investment, the firm will continue to adjust its operation mode optimally, following the optimal switching strategy.

One of the main results that we will present in this paper is the existence of an inaction region, the so-called *hysteresis region*, where the firm may be producing with losses, but waiting to take an action, which, in our setting, may be either to change its production mode or to exit permanently the market. We will show that a firm already producing and acting optimally never reaches this region. In addition, investment in this region is never optimal, under the standard assumptions of fixed investment cost and no delay (i.e., zero time-to-build). Thus, in these conditions, the relevance of the hysteresis region is questionable.

However, we may face situations where the firm can end up producing in such a region, in particular when we consider (i) investment with time-to-build and (ii) investment in undervalued firms. We study these two problems, and we provide numerical insights, regarding notably the sojourn time in the hysteresis region and how likely is that the firm will end up producing with positive profits.

In fact, many problems show that in the investment decision one may need to take into account that there is a delay in the investment, in particular in large scale (infrastructure) construction projects, as transportation infrastructure projects, power generating plant and aerospace and pharmaceutical investments. We refer to Bar-Ilan and Strange (1996) and references therein. Since the market conditions evolve during an investment lag, the profitability of the investment may also change. In our case, we are particularly interested to study how likely it is that the revenue reaches the hysteresis region.

Regarding investment in undervalued firms: in crisis periods, the revenue of firms decrease and many of them end up producing in the hysteresis region. As the recent crisis shows, business like hotels, restaurants and casinos, had severe losses, and therefore there are opportunities for private equity to buy these businesses at a valuation they could not have gotten before (examples are available at Son et al. (2020)). Thus, we consider a second situation where an equity firm has the opportunity of buying an undervalued company, which is not following an optimal strategy. We derive the investor's optimal strategy, finding the optimal time to buy the undervalued firm. We prove that, in this type of situations, it may be optimal to invest in the hysteresis region.

Both the switching and the investment problem that we address will be studied using a Real Options approach. Investment theory and real options were pioneered by Dixit and Pindyck with its seminal work Dixit and Pindyck (1994). There are many other works dealing with investment under uncertainty, namely Arrow and Fisher (1974), McDonald and Siegel (1986), and Trigeorgis et al. (1996), for instance. Authors have considered different features of investment, for instance, Huisman and Kort (2003), Pawlina and Kort (2006), and Brealey et al. (2012) address strategic options; Bjerksund and Ekern (1990), Majd and Pindyck (1987), and McDonald and Siegel (1985) discuss the option of investment; Farzin et al. (1998) and Hagspiel et al. (2016) study technology adoption problems; and Brennan and Schwartz (1985), and Myers and Majd (2001) tackle the abandonment option. Bar-Ilan and Strange (1996), and Alvarez and Keppo (2002) considered investment problems with deterministic time-to-build. In their case, they proved that an increase in the investment lag increases the investment threshold and thus delays investment. Nunes and Pimentel (2017); Couto et al. (2015) also studied a similar problem, in the context of high-speed service rails. They also considered non-constant investment costs.

Our contribution to the literature of real options is twofold. On the one hand, we analyse a switching problem in which the firm has the opportunity to switch between two production modes with different levels of risk, and, on the other hand, we present and discuss the investment strategy in a project with two alternative production modes. The literature of switching problems is vast and we refer to Duckworth and Zervos (2001), Ly Vath and Pham (2007), Guerra et al. (2018), Zervos et al. (2018), and references therein. In Zervos et al. (2018), the firm can choose between two operation modes. In mode 1, the firm is producing, whereas in mode 0 the firm stops the production, paying a running cost. A full characterisation of the optimal strategy is provided by the authors. For some parameter choices an hysteresis region is found, where the firm should produce, although a negative profit is obtained. In this region the firm waits to see what happens, before taking an irreversible decision. In Guerra et al. (2018), the authors present an economic analysis of the model and discuss the role of such a region in the investment strategy.

In the above references, the authors assume that while in mode 2 there is no production. On the contrary, in our paper we consider that in the alternative safe mode the firm may be producing, obtaining a positive profit. Additionally, we consider also the case that the firm may decide how much it has to downgrade its production.

This feature is highly relevant for seasonal or intermittent demand, common in industries as construction or tourism, or in any activity and period where an economic down-turn occurs because overall demand declines. One example comes from the airline industries. The terrorist attacks of September 11, 2001, affected the U.S. airline industry. Even with federal assistance, the losses in this industry were enormous, due to the slow rate of passenger return. In response to these losses, the major airlines cut flights and laid off their work forces, in different extend. In some cases the cuts were too large and proved to be disastrous and led these companies to bailout, whereas other, that kept a more significant work force, responded better to the crisis and managed to emerged from this crisis resilient and strong (Gittell et al. (2006)). In this paper, we show that when the firm decides to switch for a production mode less risky, it can improve their value by correctly choosing how much it should downgrade the production level.

The investment problem in alternative projects was firstly addressed by Dixit (1993). In his set up, the firm has to choose between two different projects with different sizes. The project initially chosen by the decision-maker will be active forever. The price follows a geometric Brownian motion (GBM) and the cash-flow is a linear function of the price. Additionally, the project with the larger sunk cost provides larger returns for high values of price. Assuming that p_1 and p_2 are the individual investment thresholds for each one of the projects, the main finding of Dixit's paper are the following: (i) when the initial price is small enough, the decision-maker will invest in the project with the largest option value as soon as the price reaches the smallest threshold; (ii) when the initial price is larger than the later threshold, it is optimal to invest in the price with large net present value.

This problem was also studied and extended by Décamps et al. (2006). The authors of the later paper proved that Dixit's solution is not completely correct. They prove that, for certain choices of the parameters, when the initial price is in between the two thresholds, it is optimal to wait, essentially to get more information about the price evolution, and to decide afterwards in which project to invest. Thus the optimal investment strategy is dichotomous.

Our model shares some of the features of the model of Décamps et al. (2006) and also finds that for certain range of parameters, there is an inaction region. But contrary to Décamps et al. (2006), upon investment the firm is allowed to switch back and forth the two production modes, without any *a priori* restriction on the number of times that it switches. Moreover, we allow the firm to take the exit decision in both production modes, which is also a feature that distinguish our model from the one considered in Décamps et al. (2006).

A full description of decisions that the firm may take in our model is the following:

a) Given that the firm may enter the market at any time, and may choose to enter using one of two production modes, the firm needs to decide when to invest in the market and in which production mode. This lead to a one-time decision.

b) Given that the firm is already in production, it has to decide when it is optimal to switch from a production mode to the other. This leads to a sequence of switching times.

c) The firm may decide to leave the market, and this may happen in both modes. This lead to a one-time decision.

In a nutshell, the main findings of this paper are as follows:

(a) Depending on the relationship between the parameters of the model, an hysteresis region in the more risky production mode may exist. But when the drift term is negative and small and/or the volatility is small, this region does not exist in the optimal switching strategy. These results are in line with the ones presented by Guerra et al. (2018). Moreover, increasing the switching costs increases the size of the hysteresis region, up to a certain value, after which the hysteresis region no longer exists.

(b) A firm that acts optimally and that is already producing never reaches the hysteresis region due to a continuous decrease of the revenue. We show that for particular types of investment such as investment with time-to-build or investment in undervalued firms, production in the hysteresis region may be optimal. Considering a standard investment option, the firm never invests in such a region. In Guerra et al. (2018) it is shown that this region may be attained if there is a sudden shock, that leads to a downward jump.

(c) In case the firm end up in the hysteresis region, the expected sojourn time there decreases with increasing volatility but the probability of resuming production (and switching to the safe mode) rather than exiting the market increases. This result is the opposite of the findings of Guerra et al. (2018), and can be explained by the fact that in our model in the safe mode there is still positive revenue, whereas in the model of Guerra et al. (2018) in the alternative mode (suspension) there are only running costs.

(d) Considering that the firm can choose how much it will downgrade its production, i.e. how safe is the less risky mode, then it will be willing to choose a less safe mode with increasing the drift but, on the contrary, it will choose a safer mode with increasing the uncertainty. At the same time, increasing the drift leads to a choice of a smaller value of the fixed running cost of the safe mode, which is somehow unexpected. This is explained by the fact that the sojourn time in this mode also increases, and therefore the firm wants to pay less fixed costs per unit of time. The opposite effect occurs with increasing uncertainty.

(e) Under certain conditions on the costs, it may happen that the investment region is not connected, meaning that there is an inaction region in which the decision-maker waits to see in which project to invest. The same situation occurs in Décamps et al. (2006). This inaction region exists for small values of the drift and of the volatility, but when one increases the volatility, the inaction region tends to disappear providing that the remaining parameters are fixed. However, we can see that even when the volatility is high enough there are set of parameters in which the inaction region appears. This effect is described by Décamps et al. (2006) when one allows switching from the project with the lower output flow to the one with the larger output flow.

This paper is organised as follows: in Section 2 we introduce the model and the switching and investment problems. The switching problem is then solved in Section 3, where the relevant strategies and thresholds are presented, as well as a comparative study where we assess the effects of the parameters and costs in the optimal strategy. Then, in Section 4, we assume that the safe mode can be chosen by the firm and we derive some results concerning the optimal choices of the of mode 2. Section 5 concerns the investment problem, with the presentation of the optimal investment strategy and its sensitivity with respect to the diffusion parameters. In Section 6, we discuss two particular types of investment where firms can end up producing in the hysteresis region. We also compute how likely is the firm resume production instead of exiting, and the expected time that the firm stays in the hysteresis region. Section 7 concludes the paper. In the appendixes, we provide the expressions for the parameters, and thresholds, proofs and some additional tables and figures.

2 Model

We consider a monopolistic risk neutral firm, that has the opportunity to invest in a project whose revenue evolves stochastically over time. Then, we denote by P_t the revenue at time t and we assume that $\{P_t, t \geq 0\}$ is a GBM, with drift μ and volatility $\sigma > 0$. We let r denote the risk free rate, and we assume that $r > \mu$.

The firm may operate in two alternative modes: mode 1 and mode 2. Whenever the firm is in production mode i, and the current value of the revenue is p, its instantaneous profit is π_i, with

$$\pi_i(p) = \alpha_i p - \beta_i, \quad i = 1, 2.$$

Since the power of a GMB is still a geometric Brownian motion with different drift and diffusion parameters, the results can be generalised for instantaneous profit function, as $\pi_i(p) = \alpha_i p^{\gamma_i} - \beta_i$, with $i = 1, 2$.

The coefficients β_i can be interpreted as (instantaneous) costs of production in mode i, and hence $\beta_i > 0$. Furthermore, we assume that mode 1 is more risky than mode 2, meaning that we assume the following ordering in the production parameters:

$$\alpha_1 > \alpha_2 \geq 0, \quad \beta_1 > \beta_2.$$

Moreover, the firm has the option to leave the market from both modes.

Since the firm may produce in one of the two possible modes and can abandon the market, we introduce the process: $\{Z_t, t \geq 0\}$, with $Z_t \in \{1, 2, ex\}$, where $Z_t = i$ means that the firm is operating in mode i, with $i = 1, 2$, and $Z_t = ex$ means that the firm has abandoned the market. The state ex is absorbing. For instance, a realisation of the process such that $Z_s = ex$ and $Z_t \in \{1, 2\}$, for $t > s$, is not admissible as state ex is absorbing. A *strategy* is then a realisation of the stochastic process $\{Z_t, t \geq 0\}$. We let $\mathcal{S}$ denote the set of all admissible strategies.

Considering the transition between modes, we denote the time when the j^{th} transition from state a to state b occurs by $T_j^{a,b}$, with $a, b \in \{1, 2\}$. Following Zervos et al. (2018), these times can be defined recursively by:

$$\begin{aligned} T_1^{a,b} &= \inf\{t > 0 : Z_{t^-} = a, Z_t = b\} \\ T_{j+1}^{a,b} &= \inf\{t > T_j^{a,b} : Z_{t^-} = a, Z_t = b\}, \quad a \in \{1, 2\}, \quad b \in \{1, 2, ex\}, j \in \mathbb{N}. \end{aligned}$$

The exit times are defined by

$$\tau_1 = \inf\{T_j^{1,ex} < \infty\}, \quad \tau_2 = \inf\{T_j^{2,ex} < \infty\}, \quad \tau = \inf\{\tau_1, \tau_2\}.$$

Switching from one production mode to another one implies a cost payment and leaving the market generates a cost or a salvage value. We let K_{ij} denote the transition cost from operating mode i to operating mode j, and K_x represent the exit cost (when $K_x \geq 0$) or the salvage value ($K_x < 0$). We assume that $rK_x - \beta_2 < 0$, so that for small values of revenue exit from production mode 2 may be optimal. Since $\beta_1 > \beta_2$, exit from production mode 1 may also be optimal.

If the firm is already producing, and given that currently the firm is in state $z \in \{1, 2\}$ and the initial revenue is p, the expected profit of the firm in case it follows the strategy $\mathfrak{s} \in \mathcal{S}$, which we denote by $J_{\mathfrak{s}}(z, p)$, is given by:

$$\begin{aligned} J_{\mathfrak{s}}(z, p) =& \mathbb{E}_{z,p}\Bigg[\int_0^\infty e^{-rt}\left(\pi_1(P_t)\mathcal{I}_{\{Z_t=1\}} + \pi_2(P_t)\mathcal{I}_{\{Z_t=2\}}\right)dt - \\ & -K_{12}\sum_{j=1}^{\infty} e^{-rT_j^{12}}\mathcal{I}_{\{T_j^{12}<\infty\}} - K_{21}\sum_{j=1}^{\infty} e^{-rT_j^{21}}\mathcal{I}_{\{T_j^{21}<\infty\}} - K_x e^{-r\tau}\mathcal{I}_{\{\tau<\infty\}}\Bigg], \end{aligned} \tag{1}$$

where $\mathcal{I}_A$ represents the indicator function that is equal to 1 if A holds true and 0 otherwise, and $E_{z,p}[...]$ is the expected value conditional to the information that $Z_0 = z$ and $P_0 = p$. Then the optimal switching strategy can be found by solving the optimization problem:

$$V(z, p) = \sup_{\mathfrak{s}\in\mathcal{S}} J_{\mathfrak{s}}(z, p). \tag{2}$$

Solving the switching problem above allows us to find the sequence of states (and times) of the process Z that maximises the profit of the firm, given the current level of revenue p and the operating mode z.

A firm that is not producing yet has to choose both the optimal investment time as well as the mode in which it will start operation. Afterwards, we assume that the firm is profit oriented, acting in accordance with the optimal strategy obtained in (2). At the investment moment, the firm has to pay an investment cost, which may be different according to the production process chosen. We let K_i denote the investment cost of the firm when it enters the market in mode i, with $i = 1, 2$. To facilitate the presentation, we will consider the following notation:

$$v_1(p) := V(1, p), \quad v_2(p) := V(2, p), \tag{3}$$

and

$$v^*(p) = \max\left(v_1(p) - K_1, v_2(p) - K_2\right). \tag{4}$$

Thus, the investment problem is given by

$$W(p) = \sup_{\tau \geq 0} E_p\left[e^{-r\tau} v^*(P_\tau)\right], \tag{5}$$

with E_p denoting the conditional expectation at $P_0 = p$. The investment strategy will provide us the optimal investment moment of the firm and in which production mode the firm should start operating.

3 The switching problem

In this section, we provide a complete analysis of the switching problem (2), discussing the optimal strategy, and analysing the effect of the parameters in such strategy.

In order to solve the switching problem defined in (2), we start by providing the corresponding Hamilton-Jacobi-Bellman (HJB) equations. As we have two production modes, we have two HJB equations, and each HJB equation has three members. Whenever the firm is producing in mode i, it has the following options: [1] it continues producing in that mode, [2] it switches to the other production process, or [3] it exits the market. Therefore, the associated HJB equations are coupled and are of the following form:

$$\max\{\overset{[1]}{(\mathcal{L}v_1)(p) - rv_1(p) + \pi_1(p)}, \overset{[2]}{v_2(p) - v_1(p) - K_{12}}, \overset{[3]}{-v_1(p) - K_x}\} = 0, \tag{6}$$

$$\max\{(\mathcal{L}v_2)(p) - rv_2(p) + \pi_2(p), v_1(p) - v_2(p) - K_{21}, -v_2(p) - K_x\} = 0, \tag{7}$$

where $\mathcal{L}v_i = \mu x v_i' + \frac{\sigma^2}{2}x^2 v_i''$, with v_i' and v_i'' being, respectively, the first and second derivative of v_i, with $i = 1, 2$. To simplify the explanation, we number the different decisions (producing in the same mode, switching to the other mode, and exit), using [1], [2] and [3], respectively. The HJB equations naturally divide the space into several 'action' regions, depending on where each of the parcels of the above equations is equal to zero. The theoretical framework for this problem is presented in Zervos (2003).

3.1 Optimal switching strategy

To find the solution to the HJB equations (6) and (7), we start by noticing that the ordinary differential equations that hold in region [1] (in both HJB equations) are Cauchy-Euler equations, and their solutions are as follows:

$$E_i p^{d_1} + C_i p^{d_2} + \frac{\alpha_i}{r - \mu} p - \frac{\beta_i}{r},$$

where $d_1 < 0$ and $d_2 > 1$ solve the characteristic equation $\frac{\sigma^2}{2}d^2 + (\mu - \frac{\sigma^2}{2})d - r = 0$, and E_i and C_i are constants such that the smooth-pasting conditions hold. The value function for region [3] is the value of exiting, and therefore in this region the solution is trivial and equal to $-K_x$. Finally, in region [2], the firm should optimally change its production mode from 1 to 2 or vice versa. Thus, the value function for a firm that is actually in regime i, with a revenue that belongs to region [2], is given by $v_i = v_j - K_{ij}$, with $i, j = 1, 2$ and $i \neq j$.

Depending on the set of parameters chosen, we may have different optimal strategies. Since we are considering that $rK_x - \beta_i < 0$, leaving the market will be optimal for some values of revenue. There are two optimal strategies for a firm that is already producing:

No downgrading strategy: in this case, once the firm enters production mode 1, it will never be optimal to switch to production mode 2. On the contrary, if the firm starts production in mode 2, then it will be eventually optimal to switch to mode 1, for large values of revenue. In both cases, it can be optimal to exit the market for small values of revenue. This strategy is depicted in Figure 1. In this Figure and for the rest of the paper, we use the following notation: P_{ix} is the exit threshold when the firm is in production mode i and P_{ij} is the revenue level that triggers a switch from mode i to mode j.

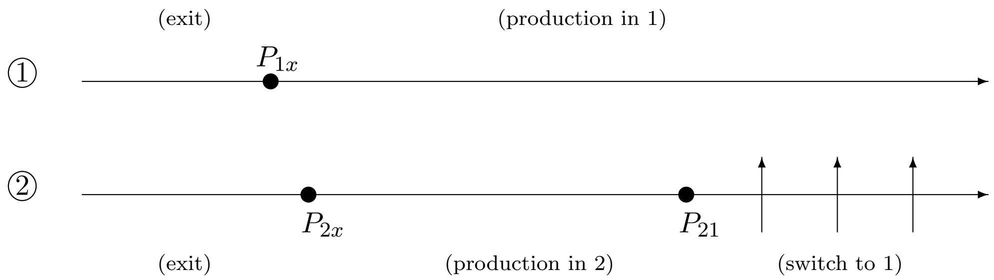


Figure 1: No downgrading strategy.

Hysteresis strategy: in contrast to the previous case, it may be optimal to switch from mode 1 to 2, and the other way around. At a first glance, we would expect an optimal strategy as the one depicted in Figure 2. However as exit is an irreversible decision, one can prove that such

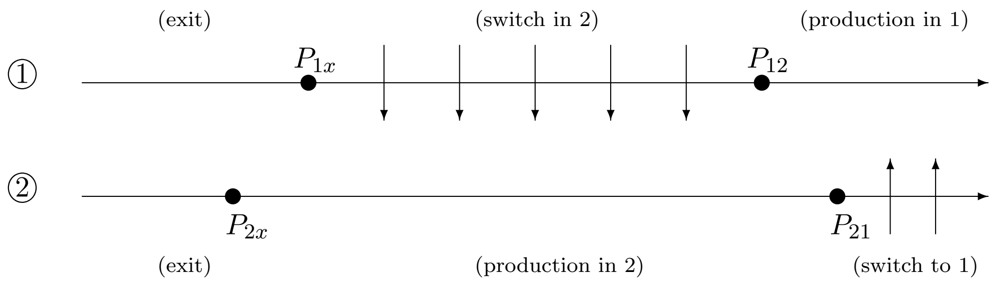


Figure 2: Downgrading without hysteresis (not optimal strategy).

strategy is not optimal. In fact, if a firm is producing in mode 1 and the revenue decreases, then it is optimal to switch to mode 2, as in this mode the firm is hedging against larger losses. In case the returns are really low, the firm is loosing money and therefore the option to exit becomes attractive. At this point, it may not be optimal to exit the market nor to switch to mode 2, since the firm pays (or receives) exactly the same in case it leaves the market either out of production mode 1 or production mode 2 (K_x), and there is a cost for switching from production mode 1 to production mode 2. Thus, it is better for the firm to wait before deciding either to switch (in case the revenue increases) or to exit (in case the revenue decreases even more), which leads to the existence of an hysteresis region. We note that in Guerra et al. (2018), the authors find also such a region, where in their case the firm may be in operating state or in mothballing.

In Figure 3 we depict this strategy, where the the hysteresis region corresponds to revenues between P_{1x} and P_h. We note that a firm will never enter the hysteresis region due to a

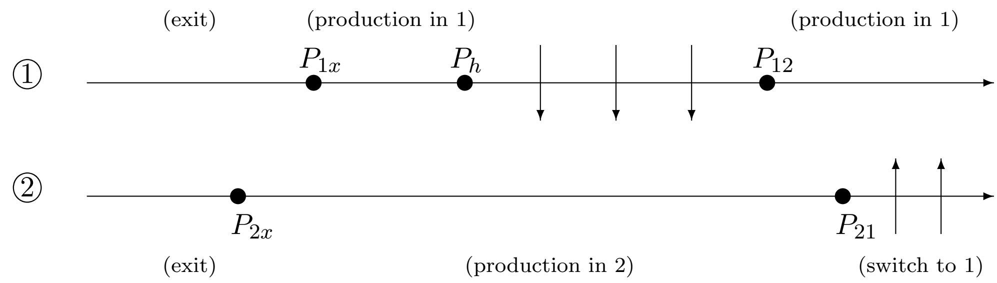


Figure 3: Hysteresis strategy.

continuous movement of the revenue. We will discuss in which situations such a region can be attained.

The optimality of the strategies depicted in Figures 1 and 3 depends on the relationship between the involved parameters. Next we present a set of conditions that will be critical for the optimality of each one of these two strategies.

Set of Conditions 1 *One of the following conditions holds true:*

i) $\beta_2 + rK_{12} \geq \beta_1$

ii) $\beta_2 + rK_{12} < \beta_1$ *and* $\pi_1(\delta) - \pi_2(\delta) \geq 0$

iii) $\beta_2 + rK_{12} < \beta_1$ *and* $\pi_1(\delta) - \pi_2(\delta) < 0$ *and* $K_{21} \geq K_{21}^{\dagger}$

iv) $\beta_2 + rK_{12} < \beta_1$ *and* $\pi_1(\delta) - \pi_2(\delta) < 0$*,* $K_{21} < K_{21}^{\dagger}$*, and* $K_{12} \geq K_{12}^{\dagger}$

where

$$\delta = \frac{(\beta_1 - rK_x)(d_2 - 1)}{\alpha_1 d_2},$$

and $K_{12}^{\dagger}$ *and* $K_{21}^{\dagger}$ *are defined in Appendix A.1.3.*

In the next proposition, we will see that when the parameters of the model satisfy the Set of Conditions 1, then the "no downgrading strategy" is optimal and the exit threshold can be computed explicitly, verifying $P_{1x} = \delta$ (the proof can be found in Appendix A). Otherwise, the "hysteresis strategy" is optimal. Indeed, condition i) states that the instantaneous cost of switching from production mode 1 to the production mode 2 adding to the running cost of producing in mode 2 is larger than the running cost of producing in mode 1. Consequently, we expect that switching from mode 1 to mode 2 is never optimal. The "no downgrading strategy" may still be optimal even if condition i) is not satisfied. If the running payoff is larger in production mode 1 than in production mode 2 for the revenue level that triggers the exit decision from mode 1, switching to mode 2 is never optimal. The same happens if the costs of switching are large enough (either $K_{21} \geq K_{21}^{\dagger}$ or $K_{12} \geq K_{12}^{\dagger}$). The same kind of bounds for the switching costs can be found in Zervos et al. (2018) and Guerra et al. (2018).

The results described above are formalised in the next proposition, where we present the value functions v_1 and v_2 associated with the previous optimal strategies. The proof of the optimality of the functions v_1 and v_2 follows the lines of the proofs provided by Zervos et al. (2018). All the parameters and thresholds are defined in Appendix A.

Proposition 1 *Consider the switching problem* (3)*. Then the "no downgrading strategy", depicted in Figure 1, is optimal if the Set Conditions 1 holds, and the value functions v_1 and v_2, defined in* (3)*, are given by the following equations:*

$$v_1(p) = \begin{cases} -K_x, & p < P_{1x} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r}, & p \geq P_{1x} \end{cases}, \tag{8}$$

$$v_2(p) = \begin{cases} -K_x, & p < P_{2x} \\ Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r}, & P_{2x} \leq p < P_{21} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} - K_{21}, & p \geq P_{21} \end{cases}. \tag{9}$$

Additionally, the exit thresholds are such that

(a) $P_{1x} = \delta$*;*

(b) $P_{1x} \leq P_{2x}$*, if* $\pi_1(\delta) \geq \pi_2(\delta)$ *or* $\Big(\pi_1(\delta) - \pi_2(\delta) < 0$ *and* $K_{21} \geq K_{21}^{\dagger}\Big)$

(c) $P_{1x} > P_{2x}$*, if* $\Big(\pi_1(\delta) - \pi_2(\delta) < 0$ *and* $K_{21} < K_{21}^{\dagger}\Big)$

If the Set of Conditions 1 does not hold, then the "hysteresis strategy", depicted in Figure 3, is optimal, and the corresponding value functions are as follows:

$$v_1(p) = \begin{cases} -K_x & p < P_{1x} \\ Ep^{d_1} + Fp^{d_2} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} & P_{1x} \leq p < P_h \\ Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r} - K_{12} & P_h \leq p < P_{12} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} & P_{12} \leq p \end{cases}, \tag{10}$$

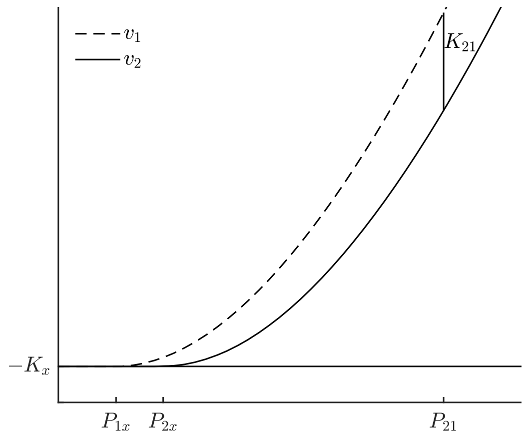
$$v_2(p) = \begin{cases} -K_x & p < P_{2x} \\ Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r} & P_{2x} \leq p < P_{21} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} - K_{21} & P_{21} \leq p \end{cases}. \tag{11}$$

The parameters and thresholds for both strategies are provided in Appendix A.

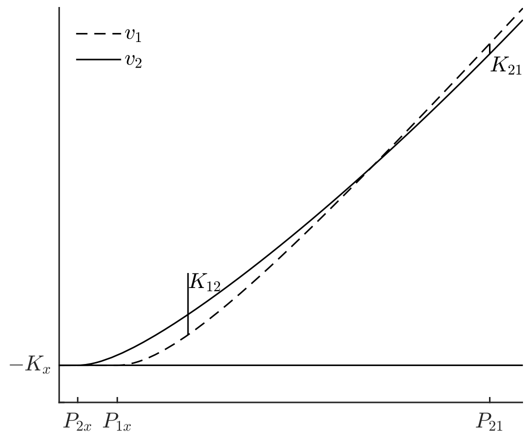
The results presented in Proposition 1 have to be interpreted as follows: given that the current revenue for the product is p, and that the firm is producing in mode $i \in \{1, 2\}$, then its value is given by $v_i(p)$. The expression for v_i depends solely on the Set Conditions 1 being satisfied (and in that case the "no downgrading strategy" is optimal) or not (for which the "hysteresis strategy" is optimal).

We note that in the "no downgrading strategy", v_1 encompass a standard exit problem, whereas for the derivation of the value function v_2, one takes into account that the firm, while in production in mode 2, has two options: the option to exit and the option to switch to mode 1. This implies that the value function in the continuation region is composed of three terms: one corresponding to the exit option, another to the switching option and, finally, the value of producing in mode 2. In the "hysteresis strategy", the value functions v_1 and v_2 are more evolved, as there are more options (and regions) available for the firm to choose.

In Figures 4 and 5 we provide an illustration of the value functions provided in Proposition 1. Figure 4 illustrates the case where it is never optimal to switch from production mode 1 to 2. The main difference between both panels is the relationship between the two exit thresholds. In panel (a) $P_{1x} \leq P_{2x}$ and, consequently, v_1 dominates v_2. Thus, for any value of $K_{12} > K_{12}^{\dagger}$, switching from 1 to 2 will never be optimal. When the firm is in production mode 2, as long as the difference between v_1 and v_2 is smaller than the switching cost K_{21}, the firm will keep



(a) $P_{1x} \leq P_{2x}$



(b) $P_{1x} > P_{2x}$

Figure 4: Value functions when the "no downgrading strategy" is optimal.

on producing in mode 2. Then, at P_{21}, $v_1(P_{21}) - v_2(P_{21}) = K_{21}$, and thus the firm switches to mode 1.

In the panel (b) we have that $P_{1x} > P_{2x}$ and, consequently, the dominance of v_1 over v_2 does not occur. But switching from production mode 1 to production mode 2, even when v_2 is larger than v_1, does not overpay the switching cost K_{12}, and, for this reason, the "no downgrading strategy" is optimal. As in panel (a), when the firm is in production mode 2 and the process hits the value P_{21}, switching to the production mode 1 is optimal because the additional gain pays the switching cost K_{21}.

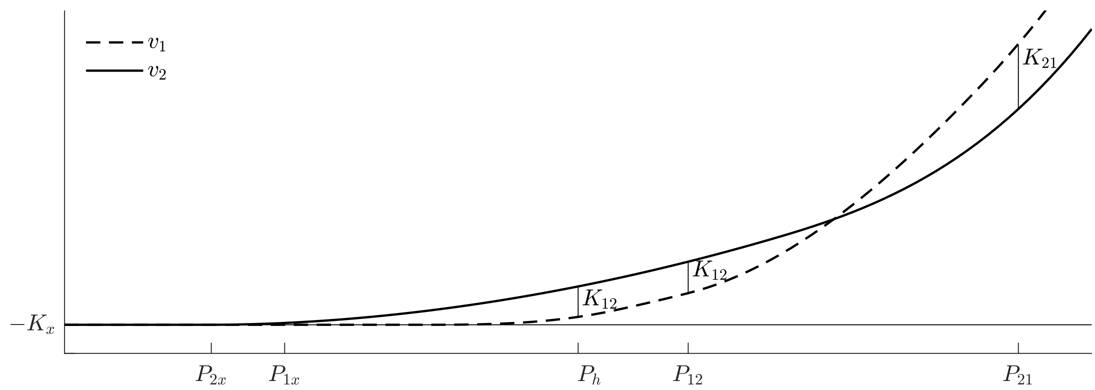


Figure 5: The hysteresis is optimal.

In Figure 5 we present an illustration of the value functions v_1 and v_2 in which the "hysteresis strategy" is optimal. When the value of producing in mode 2 is larger than the value of producing in mode 1, the firm may want to switch to production mode 2. This decision may happen at the levels of revenue P_h and P_{12} because at that points $v_2(P_h) - v_1(P_h) = v_2(P_{12}) - v_1(P_{12}) = K_{12}$. For values of $p \in (P_h, P_{12})$, we have that $v_2(p) - v_1(p) > K_{12}$, which means that the additional gain from producing in mode 2 compensates the the switching cost. When the revenue is less than P_h, switching from production mode 1 to production mode 2 is not optimal, as the profit from production mode 2 does not compensate the switching cost. So the firm keeps producing in the hysteresis region, where the revenue will be negative, waiting to decide if it should leave in case the revenue continues decreasing or if should switch to the operating mode 2, in case the revenue increases.

3.2 The effects of the parameters in the switching strategy

In this section, we assess the impact of the parameters in the optimal decisions, analysing the behaviour of the triggers and the optimality of each one of the two strategies:the "no downgrading strategy" and the "hysteresis strategy". Due to the mathematical complexity of the expressions for the triggers presented in the above subsections, this analysis will be presented numerically only. The parameters for the base case are the ones presented in Table 1.

$\mu = 0$	$\sigma = 0.2$	$r = 5\%$	$K_x = -1$	$\alpha_1 = \beta_1 = 1$	$\alpha_2 = \beta_2 = 0.5$	$K_{12} = 0.1$	$K_{21} = 0.3$

Table 1: Values for the diffusion parameters, interest rate and exit cost used along the numerical examples.

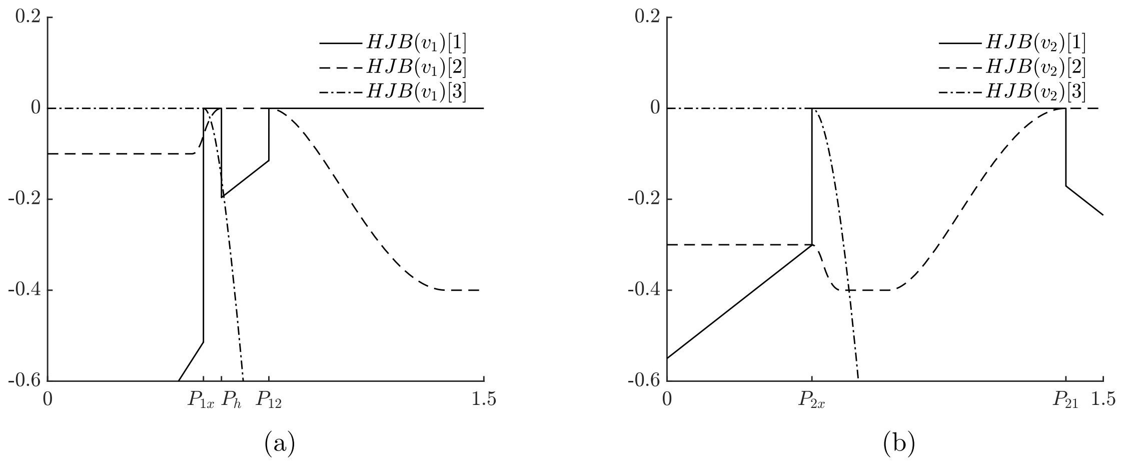


Figure 6: Numerical illustration of the verification of HJB equations (6)-(7).

Before we move to the comparative statics, in Figure 6 we show the numerical verification of the HJB equations (6)-(7) for the baseline parameters. In Figure 6, panel (a) (resp., panel (b)) we have three different lines: the solid, dashed and dash-dotted lines that represent, respectively, the first, second and third term of the HJB equation (6) (resp., (7)). The HJB equation is satisfied if all the terms are not positive and for all values of initial revenue there is one of the terms that is equal to zero. As we have already discussed, these terms are related with the production, switching and abandonment regions respectively. Thus, such regions can be identified observing the range of revenues that make each one of the terms of the HJB equation equals to zero. For instance, the solid line in panel (a) is equal to zero when $p \in (P_{1x}, P_h) \cup (P_{12,+\infty})$, which means that such region is the continuation region for a firm that is currently producing in production mode 1. Additionally, these plots guarantee that for the parameters in Table 1, the solution we provide verifies the HJB equations (6) and (7). This verification give us the guarantee that the thresholds for the baseline case are correct. The same kind of verification has been performed for all different set of values of the parameters that we present in the following subsections.

3.2.1 Comparative statics with respect to μ and σ

Next we study the impact of μ and σ on the relevant thresholds, as well as the optimal strategy.

Table 2 presents the behaviour of the thresholds with changing μ, while keeping other parameters constant, and equal to the values of the base case presented in Table 1. The last column of the table indicates the optimal strategy, with **ND** denoting the "no downgrading strategy" and **Hyst** denoting the "hysteresis strategy". We have included in Table 2 all the thresholds (note that P_h and P_{12} are not applicable for the ND case), as well as the bound on the switching cost $K_{12}^{\dagger}$. The base case (for which $\mu = 0$) corresponds to the line with grey shade.

The information regarding $K_{12}^{\dagger}$ and $K_{21}^{\dagger}$ allows us to split the situations in which the "no downgrading strategy" and "hysteresis strategy" are optimal. In fact, whenever $K_{21} \geq K_{21}^{\dagger}$ or $K_{12} \geq K_{12}^{\dagger}$, the "no downgrading strategy" is optimal. In this case, the cost of switching between modes is too expensive and therefore the exit decision is preferable when compared with switching to the production mode 2. On the contrary, in case $K_{21} < K_{21}^{\dagger}$ and $K_{12} < K_{12}^{\dagger}$, switching from mode 1 to mode 2 is a feasible option, as the associated costs are sufficiently low. In Table 2, $\beta_2 + rK_{12} < \beta_1$ and $\pi_1(\delta) - \pi_2(\delta) < 0$, thus we start by checking condition iii) in the set of conditions 1 (where only $K_{21}^{\dagger}$ has to be computed). In case condition iii) fails, then we check condition iv).

μ	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	$K_{21}^{\dagger}$	$K_{12}^{\dagger}$	Strategy
-0.250	0.973			1.019	1.712	0.001		ND
-0.150	0.931			0.965	1.565	0.014		ND
-0.030	0.729			0.694	1.397	2.625	0.037	ND
-0.010	0.619	0.706	0.771	0.575	1.389	9.725	0.161	Hyst
0.000	0.536	0.598	0.761	0.498	1.372	21.547	0.318	Hyst
0.010	0.440	0.483	0.750	0.409	1.355	59.330	0.613	Hyst
0.025	0.274	0.296	0.735	0.256	1.330		1.588	Hyst
0.030	0.215	0.232	0.729	0.202	1.322		2.167	Hyst

Table 2: Thresholds for the switching strategy with changing μ.

Table 2 suggests that increasing the drift postpones the exit decision in both production modes, which means that if the expectations about the revenue of the product increase, then the firm is more willing to stay in the market. Moreover, when the drift is larger, the firm switches from production mode 2 (the less risky) to production mode 1 (the more risky) earlier, as the switching threshold, P_{21}, decreases with μ. We also conclude from the results of Table 2 that both $K_{21}^{\dagger}$ and $K_{12}^{\dagger}$ increases with μ, and therefore the set of Conditions 1 becomes less feasible. This means that for small (and negative) values of μ the "no downgrading strategy" hysteresis strategy is optimal, since it becomes non-profitable to change from production mode 1 (risky) to production mode 2 (less risky). On the contrary, when μ is large enough the firm is less willing to exit the market, since it expects to attain large levels of revenue in the future. Thus, both switching from operating mode 1 to mode 2 and waiting in the hysteresis region may be optimal decisions. Finally, we remark that the amplitude of the hysteresis region decreases with μ. Since the firm expects larger future revenues, the hysteresis region becomes almost useless.

σ	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	$K_{21}^{\dagger}$	$K_{12}^{\dagger}$	Strategy
0.025	0.970			0.982	1.094	0.033	≈ 0	ND
0.050	0.897			0.891	1.135	0.545	0.005	ND
0.090	0.791			0.762	1.207	2.541	0.084	ND
0.100	0.765	0.834	0.839	0.732	1.225	3.290	0.106	Hyst
0.200	0.536	0.598	0.761	0.498	1.372	21.547	0.318	Hyst
0.250	0.451	0.511	0.730	0.414	1.442	48.797	0.405	Hyst
0.500	0.209	0.257	0.619	0.181	1.791		0.646	Hyst

Table 3: Thresholds for the switching strategy with changing σ.

Regarding the influence of the volatility parameter, the numerical results are presented in Table 3. We can conclude that the "hysteresis strategy" is optimal when the uncertainty is large. This happens because both $K_{21}^{\dagger}$ and $K_{12}^{\dagger}$ increase with σ, and, consequently, the set of Conditions 1 will not hold. Larger uncertainty means that the firm may wish to wait (with eventually negative returns), in the expectation that the future expected revenues will increase and cover the losses accumulated during an hysteresis period. We can also observe that the irreversible decision to exit the market (in both production modes) is postponed when the uncertainty increases. On the contrary, the firm is more willing to switch regimes. This shows us that as the market becomes less predictable, the firm tends to accommodate to the uncertainty by changing its production mode.

Before we finish this section, we note that in all the scenarios presented, a firm producing in the hysteresis region is producing at a loss since its instantaneous profit is negative. Indeed, one can easily verify that $\pi_1(P_h) = \alpha_1 P_h - \beta_1$ varies between $(-0.768, -0.294)$ (resp., $(-0.743, -0.166)$), for $\mu \in (-0.01, 0.03)$ (resp., $\sigma \in (0.1, 0.5)$). It is also interesting to notice that when either μ or σ increase, the instantaneous loss of the firm producing in the hysteresis region increases.

3.2.2 Comparative statics with respect to the switching costs K_{21} and K_{12}

We analyse the influence of the switching costs in the optimal decision. The numerical results are presented in Table 4 (for K_{12}) and Table 5 (for K_{21}).

K_{12}	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	Strategy
0.001	0.502	0.506	0.785	0.498	1.338	Hyst
0.050	0.525	0.565	0.773	0.498	1.355	Hyst
0.100	0.536	0.598	0.761	0.498	1.372	Hyst
0.315	0.563	0.716	0.718	0.500	1.430	Hyst
0.318	0.564	0.717	0.718	0.500	1.431	Hyst
0.318	0.564			0.500	1.431	ND
10.000	0.564			0.500	1.431	ND

Table 4: Thresholds for the optimal strategy with changing K_{12}.

K_{21}	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	Stat
0.300	0.536	0.598	0.761	0.498	1.372	Hyst
1.000	0.545	0.609	0.700	0.507	1.615	Hyst
1.300	0.548	0.613	0.685	0.509	1.701	Hyst
1.400	0.564			0.510	1.742	ND
5.000	0.564			0.531	2.558	ND

Table 5: Thresholds for the optimal strategy with changing K_{21}.

Observing Tables 3 and 4, we can conclude that changing either K_{12} or K_{21} leads to the same type of behaviour in the optimal strategy and thresholds. The exit thresholds in both production regimes increase when the switching costs increase, meaning that the firm is more likely to exit the market, as the switching costs are larger. As soon as the "no downgrading strategy" is optimal, one can observe that (i) K_{12} does not affect all the thresholds, because it is never optimal to switch to mode 2, and (ii) K_{21} does not affect P_{1x} only. The hysteresis threshold, P_h, and the size of the hysteresis region increase with the switching costs, meaning that the firm stays more time in such a region as the cost of switching is larger. The threshold P_{21} increases and P_{12} decreases with the switching costs, which means that switching becomes less attractive when the costs increase.

4 Optimal choice of the safe mode

In the previous sections, we have assumed that the firm cannot choose how much risk each production mode will imply, i.e., we are assuming that α_i and β_i are exogenously given. In this section, we consider the case that the investor has the option to choose amongst a set of alternative modes, with lower risk than production mode 1, quantified in the parameters α_2 and β_2. Therefore the investor needs not only to decide when to switch modes but also how much he should decrease his risk when he switches from mode 1 to mode 2.

If the current value of the process P is p and the company is operating in mode 1, we want to find the values of α_2 and β_2 that maximise its value in the current production mode, for the current value of P. Moreover, we assume that the switching costs, K_{12} and K_{21}, depend on the values of α_2 and β_2. Then, for notation purposes, we write $K_{12}(\alpha_2, \beta_2)$ and $K_{21}(\alpha_2, \beta_2)$. As these costs functions are fundamental for the analysis, we will assume an economically feasible cost structure, which will be defined further for the illustration purposes. The problem can be formalised as

$$v_1^*(p) = \max_{\substack{0<\alpha_2<\alpha_1 \\ 0<\beta_2<\beta_1}} v_1(p; \alpha_2, \beta_2).$$

Since the optimisation in both α_2 and β_2 cannot be done analytically, we proceed with a numerical illustration, optimising $v_1(p; \alpha, \beta)$ separately in β_2 and α_2. In this illustration, we consider the following scenarios: (1) β_2 is fixed and equal to 0.5; and (2) α_2 is fixed and equal to 0.816. The remaining parameters are presented in Table 1. We also present a comparative static that shows how the optimal parameters, α_2^* and β_2^*, vary for different choices of drift μ and volatility σ.

4.1 Optimal choice of α_2

Assume that the switching costs are as follows:

$$K_{12}(\alpha_2, 0.5) = \frac{\gamma_1}{(1-\alpha_2)^2} \quad \text{and} \quad K_{21}(\alpha_2, 0.5) = \gamma_2(1 - e^{\alpha_2 - 1}). \tag{12}$$

This particular choice of cost functions takes into account the following reasoning:

(i) Since the instantaneous costs β_1 and β_2 are fixed, increasing α_2 means that the production mode 2 becomes more profitable, because the instantaneous revenue becomes larger for all values of p. Thus, the switching cost K_{12} should increase with α_2;

(ii) When a firm switches from production mode 2 to mode 1, the instantaneous cost increase from β_2 to β_1. Additionally, increasing α_2 makes the slope of π_2 closer to the slope of π_1. Therefore, one can expect that K_{21} decreases with α_2.

Moreover, we choose γ_1 and γ_2 in such a way that for the parameter values given in Table 1, $K_{12}(0.5, 0.5) = 0.1$ and $K_{21}(0.5, 0.5) = 0.3$, which are the same costs as the ones considered in Section 3.2.

The costs K_{12} and K_{21}, functions of α_2, are plotted in Figure 7a. With this choice, it follows that the switching costs are comparable for low values of α_2. For large values of α_2, it is very expensive to switch from mode 1 to mode 2, but cheap to switch from mode 2 to mode 1. Panel (b) shows that there is one and only one value of α_2 that maximises v_1 when $p = 0.65 \in (P_h, P_{12})$, i.e., $v_1(0.65; \alpha_2^*, 0.5) = M$. In our example $\alpha_2^* \approx 0.816$.

In Figure 8, panels (a) and (b) show how the optimal α_2 varies with the drift μ and with the volatility σ. When μ increases the firm expects that the revenue process becomes more favourable, and hence the optimal choice leads to larger values of α_2. In opposition, in panel (b), we can observe that when the volatility increases, the firm acts more carefully and, thus, it chooses a smaller value of α_2.

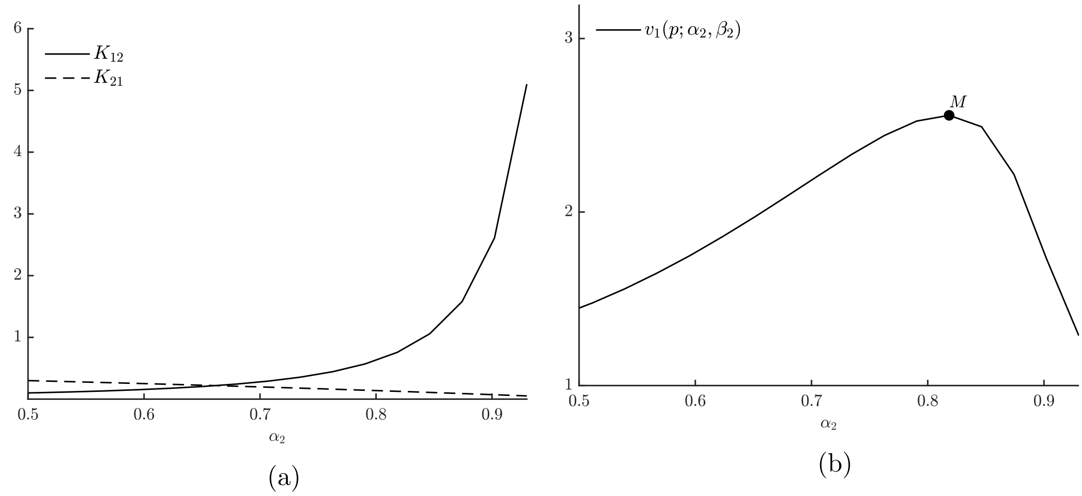


Figure 7: Panel (a): plot of K_{12} and K_{21} as functions of α_2. Panel (b): Plot of v_1 as a function of α_2. M is such that $v_1(p;\alpha_2^*,\beta_2) = M$.

4.2 Optimal choice of β_2

In this section, we analyse numerically the optimal value of β_2 when $\alpha_2 = 0.816$. We assume that the switching costs are as follows:

$$K_{12}(0.816,\beta_2) = \gamma_1 \left(\frac{1-\beta_2}{\beta_2}\right)^2 \quad \text{and} \quad K_{21}(0.816,\beta_2) = \gamma_2(\beta_2+0.3)^2. \tag{13}$$

As in the previous case, these switching costs are proposed in view of the following considerations:

(i) With α_2 fixed, when we decrease β_2 we are keeping the instantaneous profit but reducing the instantaneous cost, which increases the revenue for any initial value p. Thus K_{12} should be a decreasing function of β_2;

(ii) The switching cost K_{21} increases with β_2. This happens because when β_2 decreases, the difference between instantaneous fixed costs in the operating modes 1 and 2 becomes larger. Thus the required initial fixed cost is smaller when β_2 decreases.

We choose γ_1 and γ_2 such that for $\alpha_2 = 0.816$ and $\beta_2 = 0.5$, the costs of switching from mode 1 to mode 2 are the same as using (12), when we fix the parameters α_2 and β_2 as above.

In panel (a) of Figure 9 we illustrate the behaviour of the functions K_{12} and K_{21} with increasing β_2. We can verify that with this choice of parameters, switching from the operating mode 2 to mode 1 is considerably cheap, whereas switching from mode 1 to mode 2 entails a large cost when the instantaneous cost β_2 is small. Panel (b) shows that there exists a unique β_2 that maximises the value function $v_1(0.65;0.816,\beta_2)$. M represents the maximum of the function.

From panel (a) of Figure 10, we conclude that increasing the drift leads to a smaller optimal value for β_2, which *apriori* was not expected. In order to gain some insight about this behaviour, we computed also the exit and switching thresholds, as well as the expected time until the process either exits the market from mode 2 or switch back to mode 1 (Table 6). From this table, we conclude that increasing the drift leads to an increase in the size of the region (P_{2x}, P_{21}), which in its turn implies that the firm will stay longer producing in mode 2, which implies also that, overall, it will pay more. Hence, the firm wishes to decrease its cost per unit time, which justifies the behaviour of the optimal β_2.

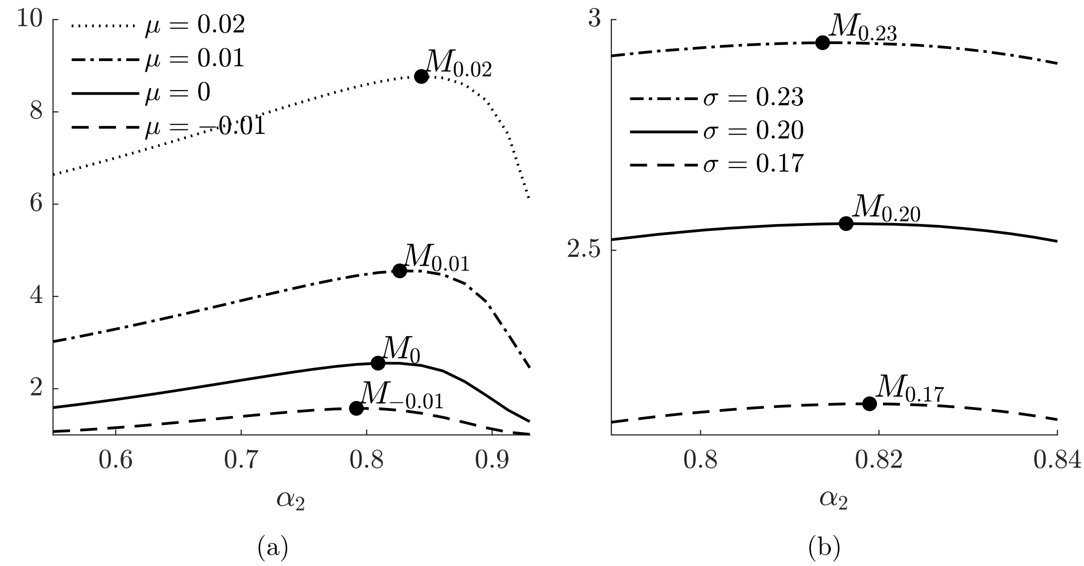


Figure 8: $v_1(0.65; \alpha_2, 0.5)$ is plotted as functions of α_2, for different values of the drift μ (resp., σ) for a given value of σ (resp., μ)), in panel (a) (resp., in panel (b)).

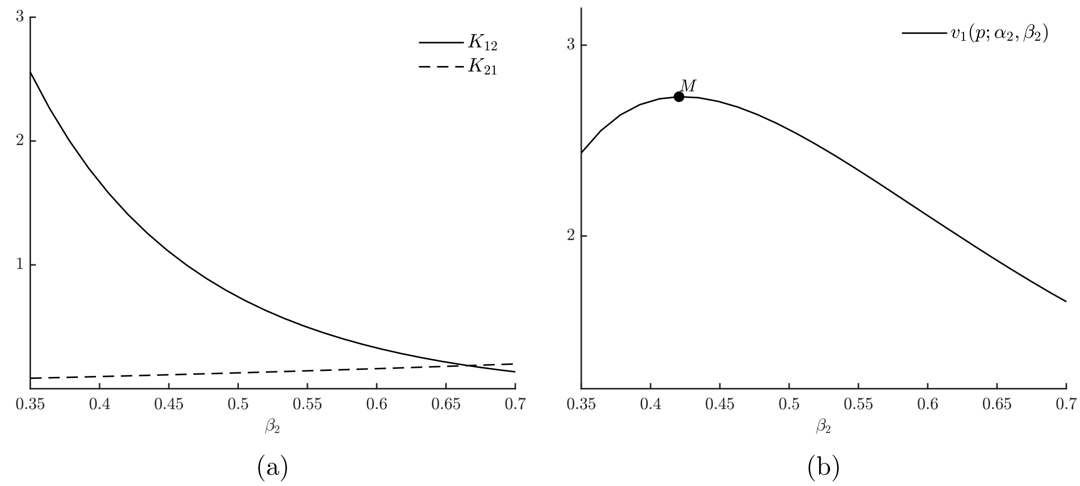


Figure 9: Panel (a): plot of K_{12} and K_{21} as a function of β_2. Panel (b): Plot of v_1 as a function of β_2. M is such that $v_1(p; \alpha_2, \beta_2^*) = M$.

μ	β_2^*	P_{2x}	P_{21}	$E[\min\{\tau_{P_{2x}}, \tau_{P_{21}}\}]$
-0.01	0.447	0.365	4.627	16.64
0.00	0.421	0.308	4.810	26.93
0.01	0.395	0.247	4.977	43.12
0.02	0.382	0.190	4.980	62.74

Table 6: Expected sojourn time of the revenue process in (P_{2x}, P_{21}) for different values of μ, when $P_0 = 0.65$. The optimal β_2 is also provided (β_2^*).

In panel (b) of Figure 10, one can observe that the optimal β_2 increases with increasing σ. Combining this information with the one presented in Table 7, we conclude that increasing σ decreases the expected time in production mode 2, and therefore the firm is willing to pay more per unit of time in this mode.

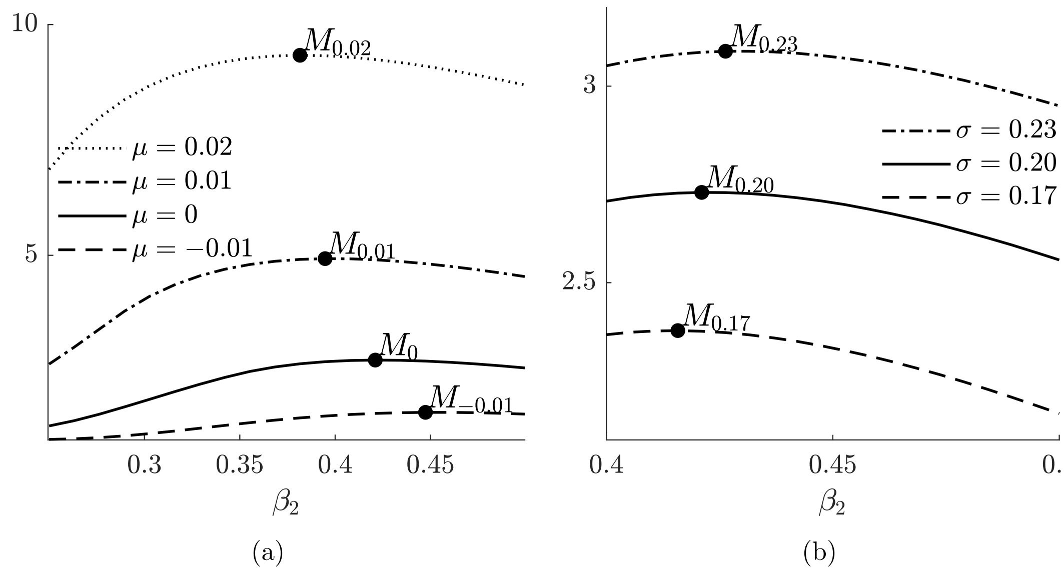


Figure 10: $v_1(0.65; 0.816, \beta_2)$ as a function of β_2, for different values of μ (panel (a)) and σ (panel (b)).

σ	β_2^*	P_{2x}	P_{21}	$E[\min\{\tau_{P_{2x}}, \tau_{P_{21}}\}]$
0.17	0.416	0.334	4.638	32.65
0.20	0.421	0.308	4.810	26.93
0.23	0.426	0.284	4.974	22.91

Table 7: Expected sojourn time of the revenue process in (P_{2x}, P_{21}) for different values of σ, when $P_0 = 0.65$. The optimal β_2 is also provided (β_2^*).

5 The investment problem

In this section, we assume that the firm is not in the market, and needs to decide when to invest and in which production mode. The investment problem is formalised in Equation (5). Standard arguments in real options (see for instance Dixit and Pindyck (1994)) allow us to conclude that the value function $W(p)$, defined as in (5), can be computed solving the HJB equation

$$\max\{(\mathcal{L}s)(p) - rs(p), -s(p) - v^*(p)\} = 0,$$

where the operator $\mathcal{L}$ is defined in the previous sections. We assume that the firm, after investment will act optimally, in accordance with the optimal switching strategy.

In this section, we will characterise the optimal investment strategy as well as its behaviours regarding the parameters of the model.

5.1 Optimal investment strategy

We assume that after investment the firm will act optimally and according with the solution of the model (2). Thus, the terminal cost, i.e. the value upon investment, denoted by v^*, is the upper envelop of v_1 and v_2 since the firm can choose any one of the two production modes.

We note that this problem is related with problem (32) of Décamps et al. (2006). But in our set-up we allow multiple switchings between mode 1 and mode 2, and vice-versa, whereas in Décamps et al. (2006) it is only allowed to switch once from mode 2 to mode 1 (in the terminology of Décamps et al. (2006), this corresponds to switch from mode 1 to mode 2).

In the rest of the paper we assume that the following set of conditions hold:

Set of Conditions 2

(i) $K_1 > -K_x$ *and* $K_2 > -K_x$

(ii) $K_1 < K_2 + K_{21}$ *and* $K_2 < K_1 + K_{12}$

From an economic point of view these two conditions are quite reasonable to impose. Condition (i) prevents instant gain, firm enters the market and immediately exits with profit Condition (ii) means that it is more costly to enter in the market with the production mode 1 (resp. 2) and to switch immediately to the production mode 2 (resp. 1) than to enter directly in mode 2 (resp. 1).

The structure of the function v^* is important in order to guess the shape of the waiting and investment regions. In the following proposition we define the function v^* in light of the value functions v_1 and v_2.

Proposition 2 *The terminal value* v^* *is given by:*

- *If* $K_1 \geq K_2$*, then*

$$v^*(p) = \begin{cases} v_2(p) - K_2, & p < z \\ v_1(p) - K_1, & p \geq z \end{cases},$$

  where

  - $z \in (P_{1x}, P_{21})$*, if the no-downgrading strategy is optimal.*
  - $z \in (P_{12}, P_{21})$*, if the hysteresis strategy is optimal.*

- *If* $K_1 < K_2$ *and* $P_{1x} > P_{2x}$*, then*

$$v^*(p) = \begin{cases} v_1(p) - K_1, & p < z_1 \\ v_2(p) - K_2, & z_1 \leq p < z_2 \\ v_1(p) - K_1, & p \geq z_2 \end{cases},$$

  where

  - $z_1 \in (P_{2x}, P_{1x})$ *and* $z_2 \in (P_{1x}, P_{21})$*, if the no-downgrading strategy is optimal.*
  - $z_1 \in (P_{2x}, P_h)$ *and* $z_2 \in (P_{12}, P_{21})$ $z \in (P_{12}, P_{21})$*, if the hysteresis strategy is optimal.*

- *Otherwise*

$$v^*(p) = v_1(p) - K_1,$$

where v_1 *and* v_2 *are given by* (8) *and* (9)*, in the no-downgrading case, and by* (10) *and* (11)*, in the hysteresis case.*

The values z, z_1 and z_2 cannot be found analytically, but, as one can see in Appendix B.1, they exist and are unique on the respective domain.

Proposition 2 shows that the terminal cost of the investment problem is highly dependent on both the investment cost and the structure of the optimal switching strategy. One can conclude that for large values of the revenue p, the perpetual value of investment in mode 1 is larger than in mode 2, regardless of the investment cost in each mode. For small values of the revenue p, the perpetual value of investment is not straightforward, since we can find situations where $v_2(p) - K_2$ dominates $v_1(p) - K_1$, and vice-versa. Based on the shape of v^*, one can guess that investment in the production mode 2 may be optimal for small values of the revenue p. Thus one expects the following strategies:

Connected investment region: The firm waits for larger revenues and then invests in the more profitable mode, which is mode 1. In this case, the value function for the investment problem (5), hereby denoted by s_1, is as follows:

$$s_1(p) = \begin{cases} B_2 p^{d_2}, & p < \gamma_3 \\ v_1(p) - K_1, & p > \gamma_3 \end{cases}. \tag{14}$$

In Figure 11, we illustrate the behaviour of s_1 as a function of the revenue p, including the relevant thresholds for the switching problem. For $p > z_1$, the value of the firm producing in

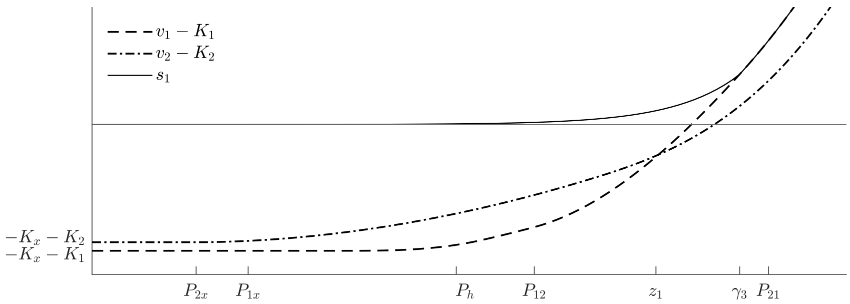


Figure 11: Plot of the value function s_1

mode 1 is larger than its value in mode 2, and therefore the investment will occur in this mode. The investment threshold, γ_3, is the value of p for which the value of investment in mode 1 is equal to the value of waiting. Thus, for values larger than this threshold, the value of investment is larger than the option to differ. We note that, in the case depicted in Figure 11, there is no complete dominance of $v_1(p) - K_1$ over $v_2(p) - K_2$. However, if $v_1(p) - K_1$ dominates $v_2(p) - K_2$, which happens when $K_1 < K_2$ and $P_{1x} \leq P_{2x}$, the optimal strategy is a threshold one, and the value function is still given by s_1 as in (14). A threshold investment decision in alternative projects were already presented by Dixit (1993).

Non-connected investment region: The firm invests for moderate values of the revenue ($p \in (\gamma_1, \gamma_2)$), and in that case it invests in the production mode 2. But when the revenue is around z_1, a point of intersection between the two curves, then it may be optimal to wait for larger values of revenue ($p \in (\gamma_3, \infty)$) and then invest in the production mode 1. Therefore, in this case, the value function for the investment problem (5) denoted by s_2 and is given by:

$$s_2(p) = \begin{cases} B_1 p^{d_2}, & p < \gamma_1 \\ v_2 - K_2, & \gamma_1 < p < \gamma_2 \\ A_2 p^{d_1} + B_2 p^{d_2}, & \gamma_2 < p < \gamma_3 \\ v_1 - K_1, & p \geq \gamma_3 \end{cases}. \tag{15}$$

In Figure 12 we plot s_2. One can observe that, for value of $p < z_1$ the value of operating in mode 2 is larger than in mode 1. Thus, as $z_1 > \gamma_1$, γ_1 triggers the investment in production mode 2. Finally for values of p larger than γ_3, the value of investment in the production mode 1 is larger than the value of investment in mode 2. Thus, γ_3 triggers investment in production mode 1. One may notice that investment in production mode 1 occurs for values of $p \in (P_{12}, P_{21})$, which means that investment in the hysteresis region is never optimal, as we state in Proposition 3. Finally, we note that in this case the optimal strategy is not a threshold type. A disconnected investment region is also find in the paper of Décamps et al. (2006). Mathematically, the inaction region found between the two investment regions is explained by the fact that v^* has an upward kink. Investing in that regions is never optimal because, there is always a solution to the equation $rs(p) - (\mathcal{L}s)(p) = 0$ that is larger than v^* and pastes conveniently v^*. The same phenomenon is described by Décamps et al. (2006).

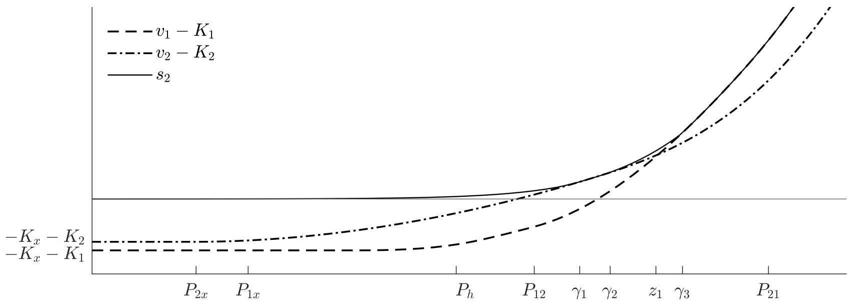


Figure 12: Plot of the value function s_2

In Figures 11 and 12 we consider that $v_1(p) - K_1$ crosses $v_2(p) - K_2$ only once. However, as one can see in Proposition 2, we may have situations where $v_1(p) - K_1$ crosses twice $v_2(p) - K_2$. Even in this case the optimal strategy is given by s_1 or s_2, depending on the set parameters.

In the next proposition, we show that the hysteresis region is never reached through investment. The attainability of this region is discussed in the next section.

Proposition 3 *Investment in the hysteresis region is never optimal.*

An immediate consequence of this proposition is the fact that a firm operating in mode 1 never exits from this mode. In fact, it is always optimal to switch to a less risky operating mode, the operating mode 2. Once in mode 2, the firm produces in a safety mode while waits to decide either to exit the market (in case the revenue decreases) or to switch to the operating mode 1 (in case the revenue increases).

Finally, in the next proposition we present the conditions for s_1 or s_2 to be the solution of the investment problem (5), which depend mainly on the relationship between costs and revenues.

Proposition 4 *Let W be the value function associated with the investment problem* (5)*. Then the following happens:*

- *If $K_1 \geq K_2$ or ($K_1 < K_2$ and $P_{1x} > P_{2x}$) then*
  - *(a) $W(p) = s_2(p)$, if $K_2 < K_2^+$ and $K_1 > K_1^-$.*
    The optimal strategy is as follows: if $p \in (\gamma_3, \infty)$, then it is optimal to invest in production mode 1, where $\gamma_3 \in (P_{12}, P_{21})$; if $p \in (\gamma_1, \gamma_2)$, then it is optimal to invest in production mode 2, where $\gamma_1 \in (P_{2x}, P_{21})$ and $\gamma_2 \in (\gamma_1, \gamma_3)$. Otherwise, the firm wait.
  - *(b) $W(p) = s_1(p)$, if $K_2 \geq K_2^+$ or $K_1 \leq K_1^-$.*
    The optimal strategy is as follows: if $p \in (\gamma_3, \infty)$, then it is optimal to invest in production mode 1, where $\gamma_3 \in (\max(P_{1x}, P_{12}), \infty)$;
- *If $K_1 < K_2$ and $P_{1x} \leq P_{2x}$, then $W(p) = s_1(p)$.*
  The optimal strategy is as follows: if $p \in (\gamma_3, \infty)$, then it is optimal to invest in production mode 1, where $\gamma_3 \in (P_{1x}, \infty)$;

The constants A_2 and B_2, the thresholds γ_1, γ_2 and γ, and the bounds K_1^- and K_2^+ are defined in Appendix A.2

5.2 The effects of the parameters in the investment strategy

In this section, we illustrate with numerical examples the results derived in the previous section. It is important to notice that when we change μ and σ, we also change the solution of the underlying switching problem (2). We will only choose costs that satisfy the Set of Conditions 2, reducing ourselves to the types of solutions, s_1 and s_2, as described in Proposition 4.

5.2.1 Comparative statics with respect to μ

In this section, we consider the parameters in Table 1, and use the investment costs $K_1 = 1.3$ and $K_2 = 1.05$. To facilitate the analysis we will use a range of values for μ also considered in Table 2. The results are shown in Table 8. The line filled in grey corresponds to the base case. We note that the strategy s_2 is optimal when $K_1 > K_1^-$ and $K_2 < K_2^+$.

Analysing the values in Table 8, one can see that when $\mu = -0.25$, the optimal switching strategy is the "hysteresis strategy" because $1.3 = K_1 > K_1^- = 1.235$ and $1.05 = K_2 < K_2^+ = 1.218$. For the remaining values of μ, $K_2 > K_2^+$ and thus the optimal strategy is described by s_1. We can conclude that when we increase the drift the firm prefers to wait and invest directly in the production mode 1 instead of investing in mode 2. This is because, as μ increases, the firm expects to attain sooner large values of revenue, and, consequently, it is more profitable to produce in mode 1. Furthermore, the switching strategy is then optimal for larger values of the drift, whereas for small (and negative) values, the firm stays in mode 1 until it eventually exit the market.

μ	K_1^-	K_2^+	γ_1	γ_2	γ_3	$\gamma_3 - \max(P_{1x}, P_{12})$	Strategy	Inv
-0.250	1.235	1.218	1.381	1.510	1.584	0.610	ND	s_2
-0.150	1.303	1.190			1.450	0.519	ND	s_1
-0.100	1.338	1.141			1.418	0.532	ND	s_1
-0.030		0.734			1.464	0.735	ND	s_1
0.000		-0.063			1.533	0.772	Hyst	s_1
0.010					1.545	0.795	Hyst	s_1
0.025					1.534	0.800	Hyst	s_1
0.030					1.521	0.791	Hyst	s_1

Table 8: Thresholds for the optimal investment strategies with changing μ. The investment costs are $K_1 = 1.3$ and $K_2 = 1.05$

The most unexpected result shown in Table 8 is the non-monotonic behaviour of the investment threshold γ_3 in the production mode 1. We note that it starts to decrease with μ, meaning that the firm invests earlier, then increases, and decreases again. This non-monotonic behaviour is also present in the distance between the investment threshold and the exit/switching thresholds (P_{1x} and P_{12}). One possible interpretation is the following: for small values of the drift (in our case, $\mu \in \{-0.25, -0.15, -0.10\}$) the investment threshold decreases with the drift, which means that as μ increases, the firm invests earlier, which agrees with standard results from real options. Note also that the distance between the investment threshold and the exit threshold P_{1x} is decreasing with the drift. If this distance would continue decreasing, then investment would take place very close to a region where exit is optimal. Thus a sudden decrease in the revenue would lead to an exit decision, which would not be optimal because there are fixed costs involved. Therefore, we see that the investment threshold changes its behaviour, and starts to increase (for $\mu \in \{-0.03, 0, 0.01\}$). Consequently the size of the region (P_{1x}, γ_3) also increases. Finally, we see another change in the behaviour, for $\mu \in \{0.025, 0.030\}$.

Since this behaviour is unexpected, in Appendix C.1 we provide a numerical verification of the HJB equations for the levels of μ where the inversion of the behaviour happens. This ensures that the numerical solution to the optimal stopping problems is correct.

In Table 8 we do not present the values of K_1^- for $\mu > -0.030$. This follows from the fact that the condition $K_1 > K_1^-$ is verified only in case the condition $K_2 < K_2^+$ (which fails for values $\mu > -0.03$). We stop computing K_2^+ as soon as it reaches negative values, since in this case $K_2 < K_2^+$ is necessarily false.

5.2.2 Comparative statics with respect to σ

In this section, we analyse the effect of the volatility in the investment strategy. In order to facilitate the numerical analysis in this case, we slightly change the base case parameters. Now we consider that $\alpha_2 = 0.6, K_{12} = 0.25, K_{21} = 0.5$, and the remaining parameters are as in Table 1. The investment costs are set as: $K_1 = 1.9$ and $K_2 = 1.5$.

From Table 9 we can conclude that the strategy s_2 is more likely to be optimal for small values of volatility. In this case the firm is willing to invest in the production mode 2 for low values of the revenue ($p \in (\gamma_1, \gamma_2)$). If the volatility increases, then the firm is sceptic to invest, and thus waits longer for large values of revenue. In this case, only investment in production mode 1 is optimal. This is highlighted by the fact that the size of the region (γ_1, γ_2) is decreasing with σ. Additionally, γ_3 increases with σ, which confirms the usual effect of increasing volatility postpones the investment decision.

In Dixit (1993) and Décamps et al. (2006), it is noticed that high levels of volatility leads to a threshold optimal strategy, meaning that it is never optimal to invest in production mode 2. This holds when the authors consider a model without a switching possibility. As one can see in this illustration, a high level of volatility does not lead necessarily to an optimal strategy s_1. If one considers a slightly larger cost K_1, for instance $K_1 = 2$, the optimal strategy would be s_2, for all levels of volatility considered in Table 9. This would happen because K_1 would be larger than K_1^-.

Since the values for the switching thresholds for this sensitivity analysis are not in Section 3.2, we present them in Appendix C.2.

σ	K_1^-	K_2^+	γ_1	γ_2	z_2	γ_3	Strategy	Inv
0.10	1.540	4.596	1.195	1.438	1.458	1.476	Hyst	s_2
0.15	1.732	4.030	1.323	1.495	1.527	1.556	Hyst	s_2
0.20	1.839	3.540	1.461	1.549	1.594	1.634	Hyst	s_2
0.24	1.892	3.197	1.578	1.593	1.649	1.696	Hyst	s_2
0.25	1.902	3.118			1.662	1.715	Hyst	s_1
0.30	1.942	2.761			1.730	1.856	Hyst	s_1

Table 9: Thresholds for the optimal investment strategies with changing σ.

5.2.3 Comparative statics with respect to the exit cost K_x

The role of the exit option in the investment decision has been studied in real options models with different features (see for instance Duckworth and Zervos (2000), Kwon (2010), Hagspiel et al. (2016)). In this section, we also analyse the impact of the exit option in the investment strategy. In fact, the exit option becomes less valuable when the exit cost increases. If $K_x \geq \beta_2/r$ then leaving the market is not optimal. For this purpose, we analyse the behaviour of the investment thresholds by increasing the value of the exit cost K_x.

We consider the parameters as in Section 5.2.2, fixing $\sigma = 0.2$. The investment and switching thresholds are summarised in Table 10. We note that the quantities P_{12}, z_2 and P_{21} do not change with K_x. Their values are: $P_{12} = 0.889$, $z_2 = 1.594$ and $P_{21} = 1.857$. We can conclude that investing in the production mode 2 is not optimal when the abandonment cost is large. Thus, the strategy s_2 is optimal only for small values of K_x.

We find that increasing the exit cost delays investment. On the one hand, when investment in the production mode 2 (with less risk) is optimal then γ_1 increases, but the remaining thresholds γ_2 and γ_3 do not change. This means that the size of the investment region in the production mode 2 decreases. The timing to invest in production mode 1 remains unchanged. On the other hand, when investment in the production mode 2 is never optimal, then γ_3 increases K_x. This means that investment in the production mode 1 is postponed.

K_x	P_{1x}	P_h	P_{2x}	K_1^-	K_2^+	γ_1	γ_2	γ_3	Strat	Inv
-1.0	0.518	0.600	0.455	1.839	3.540	1.461	1.549	1.634	Hyst	s_2
0.0	0.481	0.550	0.417	1.891	3.253	1.535	1.549	1.634	Hyst	s_2
0.2	0.473	0.541	0.409	1.899	3.199	1.548	1.549	1.634	Hyst	s_2
0.4	0.466	0.531	0.401	1.907	3.146			1.641	Hyst	s_1
1.0	0.442	0.501	0.378	1.926	2.995			1.663	Hyst	s_1

Table 10: Thresholds for the optimal investment and switching strategies with changing exit cost K_x.

6 Producing in the hysteresis region

The key feature of the hysteresis region, when it exists, is that while in that region, the firm produces at a loss (relative to safe mode), until it is optimal either to switch or to exit the market. In Section 5 we discussed the optimal strategy for a firm that intends to invest in the market and start producing at the investment moment. For this case, in Proposition 3, we proved that investment in the hysteresis region is never optimal. Moreover, from Figure 3, it is clear that the hysteresis region is never attained due to a continuous decrease of the revenue. Hence, one may wonder in which situations this region is relevant. In this section, we present two situations where the firm may attain such region.

6.1 Time-to-build

Assume that the firm invests in the market, but it only starts producing n units of time after the investment. Thus, if investment takes place at time τ, it will only be effective at time $\tau+n$, when production will start. Following the literature, the time-to-build is designated by n, with $n>0$. In this case, the investment problem can be written as follows:

$$W_n(p) = \sup_{\tau\geq 0} E_p \left[\max\left(e^{-r(\tau+n)}v_1(P_{\tau+n}) - e^{-r\tau}K_1, e^{-r(\tau+n)}v_2(P_{\tau+n}) - e^{-r\tau}K_2\right)\right] \tag{16}$$

where we use the notation W_n to emphasise the dependence of the decision on the time-to-build, which we assume to be known.

Using the strong Markov property and the law of iterated expectations, one can rewrite (16) as

$$W_n(p) = \sup_{\tau\geq 0} E_p \left[e^{-r\tau}v_n^*(P_\tau)\right] \tag{17}$$

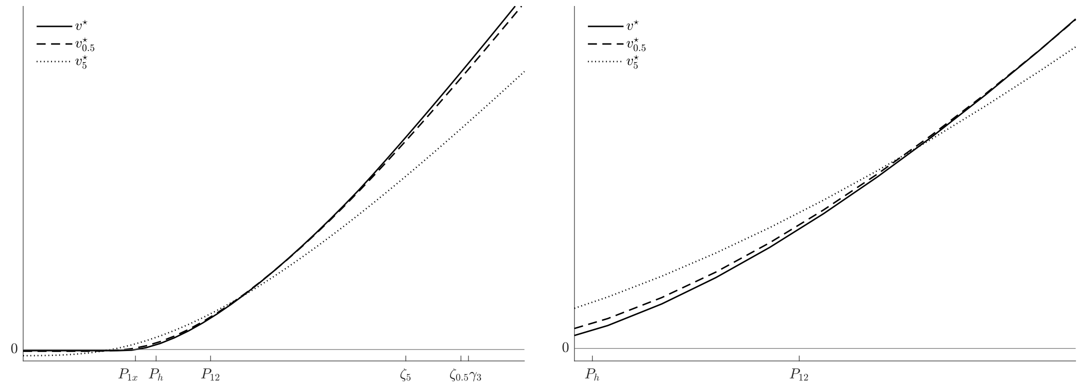
where

$$v_n^*(p) = E_p\left[\max\left(e^{-rn}v_1(P_n) - K_1, e^{-rn}v_2(P_n) - K_2\right)\right]. \tag{18}$$

The function v_n^* represents the perpetual value of the firm after investment, assuming that after investment the firm acts optimally according with the switching strategy. This perpetual value is itself an expected value, as the revenue at the moment that the firm starts operation is a random variable ($P_{\tau+n}$ is not known at time τ).

As we have seen in Section 3, the value functions v_1 and v_2 have different branches, representing the value of the firm operating in mode 1 and 2 for different values of the revenue. Therefore, in the computation of the expected value (18), one needs to take into account the probability that after n periods of time, the revenue will be in one of the branches that define the value functions v_1 and v_2. This means, in particular, that the investment strategy defined in (17) is unable to identify in which production mode the firm should optimally invest. Also as a result of the expectation operator, the function v^* is smooth enough so that the investment strategy is a threshold strategy. For notation purposes, we let ζ_n denote the investment threshold when the time-to-build is equal to n.

In Figure 13, one can see that the function v_n^* is getting smoother as n increases. Additionally, one can see that for small values of revenue $v_n^* > v^*$, but for large values of revenue $v_n^* < v^*$. This is explained by the fact that when P_τ is small there is a strictly positive probability that the revenue increases during the n periods of time, which would increase the value of $\max\left(e^{-rn}v_1(P_n) - K_1, e^{-rn}v_2(P_n) - K_2\right)$. A similar argument can be used when P_τ is large. Finally, we can observe that the investment threshold decreases when the time-to-build increases. This result is contrary to the standard results in real options, because when we increase the time-to-build, we increase the uncertainty, and, consequently, one could expect a larger investment threshold (see, for instance, Proposition 3 in Nunes and Pimentel (2017)). This does not happen because in our case the firm can invest in one of two operating modes. In the mode without time-to-build, it is optimal to invest in the production mode 1 for large values of revenue and in mode 2 for smaller values of revenue. Thus, as the perpetual value of investment in this case takes into account the probability of having either larger or smaller revenues in $\tau + n$, the threshold decreases with n.



(a) Plot of v^*, v_1^* and v_5^* for the baseline case. (b)

Figure 13: Panel (a): Plot of v^*, $v_{0.5}^*$ and v_5^* for the baseline case. Panel (b): Zoom of the figure for small values of revenue.

We present a numerical study concerning the investment threshold and the probability of entering the hysteresis region at the moment that production begins, as a function of the time-to-build, and the drift and volatility of the revenue process. We consider the values of the parameters in Table 1 and consequently the Set of Conditions 1 does not hold, meaning that the hysteresis region exists. For this section, the numerical examples were computed using the Monte-Carlo simulations for Equation (18). Thus, although the thresholds can slightly change from simulation to simulation, they allows us to confidently describe their qualitative behaviour in the sensitivity analysis.

Based on Table 11, we can conclude that the investment threshold increases with μ. In this case, the investment threshold increases because the firm wants to ensure that invests in mode 1. This result is opposite to the one found by Nunes and Pimentel (2017), where the threshold decreases with μ. This is because in the latter paper the authors consider a single project. We can also observe that the probability that the firm starts producing in the hysteresis region increases with the time-to-build and decreases with μ.

A similar analysis can be done varying the volatility. We can see that the investment threshold is monotonically increasing with the volatility for all values of n, which means that the firm postpones the investment when the market uncertainty increases. Some authors like Majd and Pindyck (1987), Milne and Whalley (2000), Bar-Ilan et al. (2002), and Nunes and Pimentel (2017) found that the monotony of the investment threshold with changing the volatility depends on the size of the time-to-build n. Such a behaviour was not found in our simulations.

Furthermore, we can easily see that changing the volatility does not result in a significant

	$\mu = -0.01$		$\mu = 0$		$\mu = 0.01$	
n	ζ_n	*Prob*	ζ_n	*Prob*	ζ_n	*Prob*
0.5	1.501	8.557×10^{-8}	1.519	3.421×10^{-11}	1.535	3.502×10^{-16}
1	1.493	0.150×10^{-3}	1.508	2.803×10^{-6}	1.510	8.912×10^{-8}
3	1.349	0.030	1.418	0.006	1.423	0.002

Table 11: Approximate values for thresholds ζ_n, for $n = 0.5, 1, 3$ for different values of μ. The remaining parameters are as in Table 1.

change in the probability that the firm starts producing in the hysteresis region. Thus, we cannot conclude about the monotony of the probability regarding the volatility.

	$\sigma = 0.1$		$\sigma = 0.2$		$\sigma = 0.25$	
n	ζ_n	*Prob*	ζ_n	*Prob*	ζ_n	*Prob*
0.5	1.298	2.479×10^{-10}	1.519	3.421×10^{-11}	1.625	5.336×10^{-11}
1	1.290	7.989×10^{-6}	1.508	2.803×10^{-6}	1.614	3.487×10^{-6}
3	1.247	0.010	1.418	0.006	1.486	0.007

Table 12: Approximate values for thresholds ζ_n, for $n = 0.5, 1, 3$ for different values of σ. The remaining parameters are as in Table 1.

Finally, these results show that the probability of investing in the hysteresis region is very small, special for small values of n. Nevertheless, this shows that this event may happen and hence the hysteresis region should be taken into account in the analysis of the investment strategy.

6.2 Pullback in valuation

In this section, we consider a private equity firm that has the opportunity of buying the company that is already operating on the market. Contrary to what have been assumed in Section 5, we consider that the investment cost is not constant: it increases with the revenue.

For illustration purposes, we assume that the investor intends to buy a firm operating in production mode 1. Thus, we let $K_1(p)$ be the purchase cost:

$$K_1(p) = v_1(p) - f(p),$$

where f can be either positive (meaning that the investment cost is in fact smaller than the value of the firm, i.e., it is sold at a discount price) or negative (in which case the investment cost is larger than the value of the firm). Hence, when $f(p) > 0$ we may interpret f as a discount, whereas in case $f(p) < 0$ we are facing a price increase.

To simplify the analysis, we assume f as follows:

$$f(p) = \begin{cases} 0, & p < P_{1x} \\ \epsilon_1, & P_{1x} \leq p \leq P_h \\ \epsilon_2, & P_h < p \leq P_{12} \\ \epsilon_3, & p > P_{12} \end{cases}$$

where $\epsilon_1 > 0, \epsilon_3 < 0$ and $\epsilon_2 \in \mathbb{R}$. This particular choice of investment costs is a result of the following reasoning:

(i) For values of the revenue in the region that leads to the hysteresis, there is a discount equal to ϵ_1 in the investment cost, and, therefore, it creates an incentive for investment, even in conditions that are not profitable *a priori*.

(ii) In the region where it is optimal to switch to the production mode 2, one can have $\epsilon_2 > 0$, meaning that there is a discount. Then, one must have $\epsilon_1 > \epsilon_2$, so that the discount is larger in the hysteresis regions.

(iii) In case $\epsilon_2 < 0$, the price is larger than the fair value of the firm. Then, $\epsilon_2 > \epsilon_3$, so that larger revenues lead to a larger increase in the investment cost.

The investment problem can be written as follows:

$$\tilde{W}(p) = \sup_{\tau>0} E\left[e^{-r\tau}\left(v_1(P_\tau) - K_1(P_\tau)\right)\right] = \sup_{\tau>0} E\left[e^{-r\tau} f(P_\tau)\right] \tag{19}$$

The investor's optimal strategy is depicted in Figure 14. In this figure, we have two cases, depending on the relationship between ϵ_1 and ϵ_2 as well as the size of the region (P_h, P_{12}). When $\epsilon_1 P_{12}^{d_1} \geq \epsilon_2 P_h^{d_1}$ (panel (a)), the value of waiting for levels of revenue in the hysteresis region (P_{1x}, P_h) is larger than the value of investment when the initial revenue is P_{12}. Thus, the investor should wait for small values of revenue. The contrary happens in panel (b). Next, we present a proposition where the value function is provided.

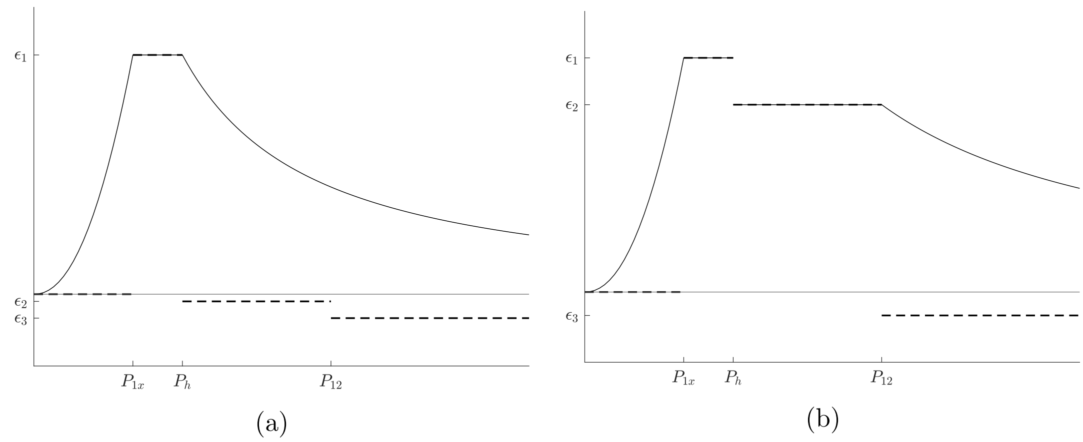


Figure 14: Panel (a): optimal strategy when $\epsilon_1 P_{12}^{d_1} \geq \epsilon_2 P_h^{d_1}$. Panel (b): optimal strategy for $\epsilon_1 P_{12}^{d_1} < \epsilon_2 P_h^{d_1}$.

Proposition 5 *Consider the investment problem defined in* (19)*. Then the value function* $\tilde{W}$ *is given by*

$$\tilde{W}(p) = \begin{cases} \tilde{B}p^{d_2}, & p \leq \tilde{\gamma}_1 \\ f_1(p), & \tilde{\gamma}_1 < p < \tilde{\gamma}_2 \\ \tilde{A}p^{d_1}, & p \geq \tilde{\gamma}_2 \end{cases}.$$

The parameters $\tilde{A}$, $\tilde{B}$, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ depend on the parameters and their expressions are provided in Appendix A.3.

6.3 Sojourn in the hysteresis region

The two situations considered before show that it is possible that the firm produces for values of the revenue that are within the hysteresis region. Then it is relevant to assess (i) how likely it is that the firm will resume production at a positive profit, and (ii) the (expected) sojourn time in this region. In order to study these two points, we consider the parameters as the ones set in Table 1, and assume that the current value of the revenue process, p, is one of the following values:

$$p_1 = P_{1x} + 0.15h, \quad p_2 = P_{1x} + 0.5h, \quad p_3 = P_{1x} + 0.85h, \quad h = \frac{P_h - P_{1x}}{2}. \tag{20}$$

Thus, p_1 is a point close to the exit threshold, p_2 is half way in the hysteresis region, and p_3 is close to the threshold P_h, where the firm leaves the hysteresis region and start producing with a larger revenue in production mode 2.

In order to study (i), we note that since the revenue follows a geometric Brownian motion, such probability has the following expression

$$Pr_i = Pr\left\{\tau_{P_{1x}} > \tau_{P_h} | P_0 = p_i\right\} = \begin{cases} \frac{\left(\frac{p_i}{P_{1x}}\right)^{1-\frac{2\mu}{\sigma^2}} - 1}{\left(\frac{P_h}{P_{1x}}\right)^{1-\frac{2\mu}{\sigma^2}} - 1}, & \frac{\sigma^2}{2} - \mu > 0 \\ \frac{1-\left(\frac{p_i}{P_{1x}}\right)^{1-\frac{2\mu}{\sigma^2}}}{1-\left(\frac{P_h}{P_{1x}}\right)^{1-\frac{2\mu}{\sigma^2}}}, & \frac{\sigma^2}{2} - \mu < 0 \end{cases}, \tag{21}$$

for $i = 1, 2, 3$, where $\tau_{P_{1x}}$ is the exit time (from production mode 1) and τ_{P_h} is the time at which the process leaves the hysteresis region, and switches to production mode 2.

To analyse point (ii), we use the following expression for the the expected time that the firm stays in the hysteresis region:

$$Est_i = E[\min\{\tau_{P_{1x}}, \tau_{P_h}\}] = \begin{cases} \frac{1}{\frac{\sigma^2}{2} - \mu}\left[\log \frac{p_i}{P_{1x}} - \log \frac{P_{1x}}{P_h} Pr_i\right], & \frac{\sigma^2}{2} - \mu > 0 \\ \infty, & \frac{\sigma^2}{2} - \mu < 0 \end{cases}. \tag{22}$$

The expressions for these quantities can be found, for instance, in Section of 15.3.6 Karlin and Taylor (1981).

In Table 13, we study (i) and (ii) as functions of the diffusion parameter μ. We can see that all the probabilities of leaving the hysteresis region by resuming production in mode 2 increase with μ. On the contrary, the expected sojourn time in the hysteresis region decreases because the size of the hysteresis region is also decreasing. Furthermore, for fixed μ, $Pr_1 < Pr_2 < Pr_3$ and $Est_1 > Est_2 > Est_3$. Since we are considering values of the process closer to the threshold P_h, it becomes more likely to leave the hysteresis region by hitting this bound than by exiting the market. Additionally, the expected sojourn time in hysteresis decreases when the revenue p gets closer to P_h. These results are not surprising, because as we increase the drift, it is more likely that the revenue increases, and, therefore, the firm will leave the hysteresis region earlier and will start producing in mode 2.

μ	Pr_1	Pr_2	Pr_3	Est_1	Est_2	Est_3
-0.010	0.146	0.492	0.846	0.056	0.107	0.053
0.000	0.150	0.500	0.850	0.039	0.075	0.037
0.010	0.153	0.506	0.853	0.029	0.055	0.028

Table 13: Impact of the sojourn in hysteresis as a function of μ when $\sigma = 0.20$.

In Table 14, we analyse (i) and (ii) as functions of the volatility. We also add information regarding the thresholds P_{1x}, P_h, which allows us to understand better the results. As we saw in Section 3.2.1, the two thresholds and the size of the hysteresis region decrease with the volatility. The probabilities that the firm leaves the hysteresis region by starting production in mode 2 slightly decrease with the volatility. Additionally, the expected time in this region decreases. This suggests that the firm takes a decision of leaving the hysteresis region sooner with increasing volatility. Looking at the probabilities, it is likely that the firm leaves this region by abandoning the market, which is an interesting result, specially in view of the decreasing thresholds.

σ	P_{1x}	P_h	Pr_1	Pr_2	Pr_3	Est_1	Est_2	Est_3
0.200	0.440	0.483	0.153	0.506	0.853	0.029	0.055	0.028
0.300	0.310	0.352	0.152	0.504	0.852	0.024	0.045	0.022
0.500	0.167	0.204	0.151	0.502	0.851	0.021	0.039	0.010

Table 14: Impact of the sojourn in hysteresis as a function of σ when $\mu = 0.01$.

7 Conclusions

In this paper, we study the effect of having alternative production modes in the strategy of a company, not only in terms of the actual operating mode but also in terms of the investment strategy. We analyse this question by solving a switching problem and an investment problem. In the switching problem, we find the optimal sequence of operating modes as a function of the revenue, for a firm that is already producing. Here, exiting the market is part of the optimal strategy. In the investment problem, we provide the optimal time to invest as well as in which project the firm should invest.

Regarding the switching problem, we prove that, depending on the parameters, there are only two optimal strategies: the "no downgrading strategy" and "the hysteresis strategy". In the "no downgrading strategy" the firm will only switch from the safe mode to the risky mode or exit the market while in the "the hysteresis strategy" the firm will be able to switch between the two modes. In this case, there is a region where the firm may stay in the market producing in the riskier mode, with a possible negative return, waiting to see if the conditions of the market improve (and in that case it changes to the less risky mode) or deteriorate (and in that case it leaves the market). We also study the impact of the costs and the parameters of the underlying process. Increasing the drift and/or the volatility makes the hysteresis case more interesting for the firm, whereas increasing the switching costs has the opposite effect.

Motivated by the current economic situation where companies need to decide on lay-off regimes, for example, we consider the optimal choice of the safe production mode. To formalize the problem, we propose a structure to the switching costs as functions of the parameters α_2 and β_2. We consider some numerical situations and we conclude that, in fact, the choice of the alternative mode impacts the profitability of the firm. The optimal values of α_2 and β_2 are monotone with μ and σ.

Then we study the investment problem, where we define not only the optimal investment threshold but also the optimal production mode in which the firm has to start producing. Similarly to Décamps et al. (2006), we show that the investment region may be disconnected, $(\gamma_1, \gamma_2) \cup (\gamma_3, \infty)$, for certain set of parameters. This happens because the value of waiting when the initial revenue belongs to (γ_2, γ_3) is larger than the perpetual value of investment. We show that when we increase the drift, the firm prefers to wait and invest directly in the production mode 1 instead of investing in mode 2. But the most interesting finding is that the investment threshold is not monotonic as a function of the drift. When the volatility increases, the optimal strategy changes and it start being optimal to invest in the safer mode 2 for small values of revenue. The opposite happens with increasing exit costs, since the firm prefers to wait to invest in production mode 1.

Finally, we conclude that in the setup that we are proposing, the hysteresis region does not have a special interest, as it is never reached. But we describe two other models where the firm may find itself producing in this region. In the first situation, we assume that there is a delay in the investment decision and in the second case, we consider an investor that intends to buy a undervalued firm. In the model with time-to-build, we find that the investment threshold increases with both μ and σ.

A Parameters and thresholds

In this appendix, we present the smooth pasting conditions for the optimisation problems (2) and (5), which allows to derive the constant terms and the thresholds.

A.1 Switching problem (2)

As discussed in Section 3, according to the relationship between the parameters, there are two optimal strategies. A technical analysis of a similar switching problem can be seen in Zervos et al. (2018).

A.1.1 "No downgrading" strategy

For the "no downgrading" strategy, the smooth-fit conditions applied to the thresholds P_{1x}, P_{2x} and P_{21} are

$$0 = AP_{1x}^{d_1} + \frac{\alpha_1}{r-\mu}P_{1x} - \frac{\beta_1}{r} + K_x \tag{23}$$

$$0 = d_1 AP_{1x}^{d_1} + \frac{\alpha_1}{r-\mu}P_{1x} \tag{24}$$

$$0 = CP_{2x}^{d_1} + DP_{2x}^{d_2} + \frac{\alpha_2}{r-\mu}P_{2x} - \frac{\beta_2}{r} + K_x \tag{25}$$

$$0 = Cd_1 P_{2x}^{d_1} + Dd_2 P_{2x}^{d_2} + \frac{\alpha_2}{r-\mu}P_{2x} \tag{26}$$

$$0 = (C-A)P_{21}^{d_1} + DP_{21}^{d_2} + \frac{\alpha_2-\alpha_1}{r-\mu}P_{21} - \frac{\beta_2-\beta_1}{r} + K_{21} \tag{27}$$

$$0 = (C-A)d_1 P_{21}^{d_1} + Dd_2 P_{21}^{d_2} + \frac{\alpha_2-\alpha_1}{r-\mu}P_{21} \tag{28}$$

Solving equations (23)-(24) allow us to get

$$A = -\frac{\alpha_1}{d_1(r-\mu)}P_{1x}^{1-d_1} \tag{29}$$

$$P_{1x} = \frac{d_2-1}{\alpha_1 d_2}(\beta_1 - rK_x) = \delta. \tag{30}$$

Computing Equation (26), minus d_1 and multiplied by Equation (25), leads to:

$$D = -\frac{d_1 P_{2x}^{-d_2}}{(d_2-d_1)r}\left[\frac{-d_2\alpha_2}{d_2-1}P_{2x} + (\beta_2 - rK_x)\right]. \tag{31}$$

Performing (26) minus d_2 multiplied by (25)

$$C = -\frac{d_2 P_{2x}^{-d_1}}{(d_2-d_1)r}\left[\frac{d_1\alpha_2}{d_1-1}P_{2x} + (-\beta_2 + rK_x)\right]. \tag{32}$$

Calculating $[(28) - d_1(27)]$

$$D = -\frac{d_1 P_{21}^{-d_2}}{(d_2-d_1)r}\left[\frac{d_2(\alpha_1-\alpha_2)}{d_2-1}P_{21} + (\beta_2 - \beta_1 - rK_{21})\right]. \tag{33}$$

Similarly, Equation (28) and Equation (27) multiplied d_1 can be simplified allowing us to get

$$C - A = -\frac{d_2 P_{21}^{-d_1}}{(d_2-d_1)r}\left[-\frac{d_1(\alpha_1-\alpha_2)}{d_1-1}P_{21} - (\beta_2 - \beta_1 - rK_{21})\right]. \tag{34}$$

Taking into account that the parameter D is given by the expressions (31),(33) and C by the expressions (29),(32), (34), we can find expressions for boundary points P_{21} and P_{2x}. In fact, $P_{21} > P_{2x}$ satisfy the system of equations

$$\begin{aligned} G_1(P_{21}, P_{2x}) := P_{21}^{-d_1} &\left[\frac{(\alpha_2-\alpha_1)(1-d_2)}{r-\mu}P_{21} + d_2\left(\frac{\beta_2-\beta_1}{r} - K_{21}\right)\right] \\ &- A(d_1-d_2) - P_{2x}^{-d_1}\left[\frac{\alpha_2(1-d_2)}{r-\mu}P_{2x} + d_2\left(\frac{\beta_2}{r} - K_x\right)\right] = 0 \end{aligned} \tag{35}$$

$$\begin{aligned} G_2(P_{21}, P_{2x}) := P_{2x}^{-d_2} &\left[\frac{\alpha_2(1-d_1)}{r-\mu}P_{2,ex} + d_1\left(\frac{\beta_2}{r} - K_x\right)\right] \\ &- P_{21}^{-d_2}\left[\frac{(\alpha_2-\alpha_1)(1-d_1)}{r-\mu}P_{21} + d_1\left(\frac{\beta_2-\beta_1}{r} - K_{21}\right)\right] = 0. \end{aligned} \tag{36}$$

A.1.2 "Hysteresis" strategy

Applying the smooth-fit conditions to the thresholds $P_{2x}, P_{1x}, P_h, P_{12}$ and P_{21} we get the following system of equations:

$$0 = EP_{1x}^{d_1} + FP_{1x}^{d_2} + \frac{\alpha_1}{r-\mu}P_{1x} - \frac{\beta_1}{r} + K_x \tag{37}$$

$$0 = d_1EP_{1x}^{d_1} + d_2FP_{1x}^{d_2} + \frac{\alpha_1}{r-\mu}P_{1x} \tag{38}$$

$$0 = (E-C)P_h^{d_1} + (F-D)P_h^{d_2} + \frac{\alpha_1-\alpha_2}{r-\mu}P_h - \frac{\beta_1-\beta_2}{r} + K_{12} \tag{39}$$

$$0 = d_1(E-C)P_h^{d_1} + d_2(F-D)P_h^{d_2} + \frac{\alpha_1-\alpha_2}{r-\mu}P_h \tag{40}$$

$$0 = (C-A)P_{12}^{d_1} + DP_{12}^{d_2} + \frac{\alpha_2-\alpha_1}{r-\mu}P_{12} - \frac{\beta_2-\beta_1}{r} - K_{12} \tag{41}$$

$$0 = d_1(C-A)P_{12}^{d_1} + d_2DP_{12}^{d_2} + \frac{\alpha_2-\alpha_1}{r-\mu}P_{12} \tag{42}$$

$$0 = CP_{2x}^{d_1} + DP_{2x}^{d_2} + \frac{\alpha_2}{r-\mu}P_{2x} - \frac{\beta_2}{r} + K_x \tag{43}$$

$$0 = d_1CP_{2x}^{d_1} + d_2DP_{2x}^{d_2} + \frac{\alpha_2}{r-\mu}P_{2x} \tag{44}$$

$$0 = (C-A)P_{21}^{d_1} + DP_{21}^{d_2} + \frac{\alpha_2-\alpha_1}{r-\mu}P_{21} - \frac{\beta_2-\beta_1}{r} + K_{21} \tag{45}$$

$$0 = d_1(C-A)P_{21}^{d_1} + d_2DP_{21}^{d_2} + \frac{\alpha_2-\alpha_1}{r-\mu}P_{21} \tag{46}$$

Taking into account the relationships:

$$r = -\frac{\sigma^2}{2}d_1d_2, \quad \mu = \frac{\sigma^2}{2}(1-d_1-d_2) \quad \text{and} \quad r-\mu = -\frac{\sigma^2}{2}(1-d_1)(1-d_2)$$

we can obtain

$$\frac{r(d_2-1)}{d_2(r-\mu)} = -\frac{d_1}{1-d_1} = \frac{d_1}{d_1-1} \qquad \frac{r(d_1-1)}{d_1(r-\mu)} = -\frac{d_2}{1-d_2} = \frac{d_2}{d_2-1}.$$

Computing $d_1(41) - (42)$ we get

$$D = -\frac{d_1P_{12}^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2(\alpha_2-\alpha_1)}{d_2-1}P_{12} + (\beta_1-\beta_2-rK_{12})\right]. \tag{47}$$

Simplifying the equations (42) and d_2 multiplied by (41), we obtain

$$C - A = -\frac{d_2 P_{12}^{-d_1}}{(d_2 - d_1)r} \left[\frac{d_1(\alpha_2 - \alpha_1)}{d_1 - 1} P_{12} + (\beta_1 - \beta_2 - rK_{12}) \right]. \tag{48}$$

Analysing Equations (45), (46), we get

$$D = -\frac{d_1 P_{21}^{-d_2}}{(d_1 - d_2)r} \left[\frac{d_2(\alpha_2 - \alpha_1)}{d_2 - 1} P_{21} + (\beta_1 - \beta_2 + rK_{21}) \right] \tag{49}$$

$$C - A = -\frac{d_2 P_{21}^{-d_1}}{(d_2 - d_1)r} \left[\frac{d_1(\alpha_2 - \alpha_1)}{d_1 - 1} P_{21} + (\beta_1 - \beta_2 + rK_{21}) \right]. \tag{50}$$

Since we have two expressions for D and to $C - A$, we are able to define the equations that allow us to compute the thresholds P_{21} and P_{12}

$$P_{21}^{-d_1} \left[\frac{d_1(\alpha_2 - \alpha_1)}{d_1 - 1} P_{21} + (\beta_1 - \beta_2 + rK_{21}) \right] - P_{12}^{-d_1} \left[\frac{d_1(\alpha_2 - \alpha_1)}{d_1 - 1} P_{12} + (\beta_1 - \beta_2 - rK_{12}) \right] = 0$$

$$P_{21}^{-d_2} \left[\frac{d_2(\alpha_2 - \alpha_1)}{d_2 - 1} P_{21} + (\beta_1 - \beta_2 + rK_{21}) \right] - P_{12}^{-d_2} \left[\frac{d_2(\alpha_2 - \alpha_1)}{d_2 - 1} P_{12} + (\beta_1 - \beta_2 - rK_{12}) \right] = 0.$$

Taking into account Equations (37), (38) as well as (37) and (38), we obtain

$$F = -\frac{d_1 P_{1x}^{-d_2}}{(d_1 - d_2)r} \left[\frac{d_2 \alpha_1}{d_2 - 1} P_{1x} + (-\beta_1 + rK_x) \right] \tag{51}$$

$$F - D = -\frac{d_1 P_h^{-d_2}}{(d_1 - d_2)r} \left[\frac{d_2(\alpha_1 - \alpha_2)}{d_2 - 1} P_h + (-\beta_1 + \beta_2 + rK_{12}) \right] \tag{52}$$

Combining Equations (51)-(52) with (47)

$$\begin{aligned} 0 = P_{1x}^{-d_2} \left[\frac{d_2 \alpha_1}{d_2 - 1} P_{1x} + (-\beta_1 + rK_x) \right] - -P_h^{-d_2} \left[\frac{d_2(\alpha_1 - \alpha_2)}{d_2 - 1} P_h + (-\beta_1 + \beta_2 + rK_{12}) \right] - \\ - P_{12}^{-d_2} \left[\frac{d_2(\alpha_2 - \alpha_1)}{d_2 - 1} P_{12} + (\beta_1 - \beta_2 - rK_{12}) \right] \end{aligned} \tag{53}$$

A different expression for D can be obtained solving the equations (43) and (44):

$$D = -\frac{d_1 P_{2x}^{-d_2}}{(d_1 - d_2)r} \left[\frac{d_2 \alpha_2}{d_2 - 1} P_{2x} + (-\beta_2 + rK_x) \right]. \tag{54}$$

Combining (54) using (47) multiplied by -1, we get

$$P_{2x}^{-d_2} \left[\frac{d_2 \alpha_2}{d_2 - 1} P_{2x} + (-\beta_2 + rK_x) \right] - P_{12}^{-d_2} \left[\frac{d_2(\alpha_2 - \alpha_1)}{d_2 - 1} P_{12} + (\beta_1 - \beta_2 - rK_{12}) \right] = 0. \tag{55}$$

Solving Equations (37), (38), and (43), (44), we obtain

$$E = -\frac{d_2 P_{1x}^{-d_1}}{(d_2 - d_1)r} \left[\frac{d_1 \alpha_1}{d_1 - 1} P_{1x} + (-\beta_1 + rK_x) \right] \tag{56}$$

$$C = -\frac{d_2 P_{2x}^{-d_1}}{(d_2 - d_1)r} \left[\frac{d_1 \alpha_2}{d_1 - 1} P_{2x} + (-\beta_2 + rK_x) \right] \tag{57}$$

From (39) and (40), we can compute

$$E - C = -\frac{d_2 P_h^{-d_1}}{(d_2 - d_1)r} \left[\frac{d_1(\alpha_1 - \alpha_2)}{d_1 - 1} P_h + (-\beta_1 + \beta_2 + rK_{12}) \right]. \tag{58}$$

Therefore,

$$P_{1x}^{-d_1}\left[\frac{d_1\alpha_1}{d_1-1}P_{1x}+(-\beta_1+rK_x)\right]-P_{2x}^{-d_1}\left[\frac{d_1\alpha_2}{d_1-1}P_{2x}+(-\beta_2+rK_x)\right]$$
$$-P_h^{-d_1}\left[\frac{d_1(\alpha_1-\alpha_2)}{d_1-1}P_h+(-\beta_1+\beta_2+rK_{12})\right]=0 \quad (59)$$

Equations (53)-(55)-(59) allow us to recover thresholds P_{1x}, P_{2x} and P_h.

A.1.3 Constants $K_{12}^{\dagger}$ and $K_{21}^{\dagger}$

Looking at the definition of the functions G_1 and G_2 defined in (35) and (36), we know that these functions depend on K_{21}. To highlight such a dependence, we write $G_1(P_{21}, P_{2x}) \equiv G_1(P_{21}, P_{2x}; K_{21})$ and $G_2(P_{21}, P_{2x}) \equiv G_2(P_{21}, P_{2x}; K_{21})$. Then, we proceed as Zervos et al. (2018), to find the bounds $K_{12}^{\dagger}$ and $K_{21}^{\dagger}$.

The $K_{21}^{\dagger}$ is such that there is a unique solution $(x, y, k) = (x, y, K_{21}^{\dagger})$, with $y > x$, to the system of equations

$$G_1(x,y,k)=0, \quad G_2(x,y,k)=0, \quad G_1(\delta,y,k)=0,$$

where δ is defined in (30). The bound for K_{12} is

$$K_{12}^{\dagger}=-K_{21}+\frac{\hat{x}^{d_2}}{r}\left[P_{21}^{-d}\left(\frac{(\alpha_1-\alpha_2)d_2}{d_2-1}P_{21}-(\beta_1-\beta_2+rK_{21})\right)-\right.$$
$$\left.\hat{x}^{-d}\left(\frac{(\alpha_1-\alpha_2)d_2}{d_2-1}\hat{x}-(\beta_1-\beta_2+rK_{21})\right)\right], \quad (60)$$

where $\hat{x} \in [P_{2x}, P_{21}]$ is a solution to:

$$(\alpha_2-\alpha_1)x\left[\frac{d_1}{d_1-1}-\frac{d_2}{d_2-1}\right]+x^{d_1}P_{21}^{-d_1}\left[\frac{(\alpha_1-\alpha_2)d_1}{d_1-1}P_{21}-(\beta_1-\beta_2+rK_{21})\right]-$$
$$x^{d_2}P_{21}^{-d_2}\left[\frac{(\alpha_1-\alpha_2)d_2}{d_2-1}P_{21}-(\beta_1-\beta_2+rK_{21})\right]=0. \quad (61)$$

Note that $K_{21}^{\dagger}$ is independent of K_{12} and K_{21}, but $K_{12}^{\dagger}$ depends on K_{21}.

A.2 Investment Problem 5

In this section, we will present the smooth-pasting conditions to find the parameters and thresholds associated to the Investment Problem defined in equation 5.

A.2.1 $K_2^+ \geq K_2$ and $K_1 \geq K_1^-$

We start by noticing that the function s_2 can be written as

$$v^*(p)=\begin{cases} B_1p^{d_2} & p\in[0,\gamma_1) \\ Cp^{d_1}+Dp^{d_2}+\frac{\alpha_2}{r-\mu}p-\frac{\beta_2}{r}-K_2 & p\in[\gamma_1,\gamma_2] \\ A_1p^{d_1}+B_2p^{d_2} & p\in(\gamma_2,\gamma_3) \\ Ap^{d_1}+\frac{\alpha_1}{r-\mu}p-\frac{\beta_1}{r}-K_1 & p\in[\gamma_3,+\infty). \end{cases} \quad (62)$$

Using the smooth-fit conditions, the parameters B_1, A_1 and A_2, and the thresholds γ_1, γ_2 and γ_3 satisfy the following equations:

$$\begin{aligned}
0 &= (B_1 - D)\gamma_1^{d_2} - C\gamma_1^{d_1} - \frac{\alpha_2}{r-\mu}\gamma_1 + \frac{\beta_2}{r} + K_2 \\
0 &= (B_1 - D)d_2\gamma_1^{d_2} - Cd_1\gamma_1^{d_1} - \frac{\alpha_2}{r-\mu}\gamma_1 \\
0 &= (C - A_1)\gamma_2^{d_1} + (D - B_2)\gamma_2^{d_2} + \frac{\alpha_2}{r-\mu}\gamma_2 - \frac{\beta_2}{r} - K_2 \\
0 &= (C - A_1)d_1\gamma_2^{d_1} + (D - B_2)d_2\gamma_2^{d_2} + \frac{\alpha_2}{r-\mu}\gamma_2 \\
0 &= (A - A_1)\gamma_3^{d_1} - B_2\gamma_3^{d_2} + \frac{\alpha_1}{r-\mu}\gamma_3 - \frac{\beta_1}{r} - K_1 \\
0 &= (A - A_1)d_1\gamma_3^{d_1} - B_2d_2\gamma_3^{d_2} + \frac{\alpha_1}{r-\mu}\gamma_3
\end{aligned}$$

Solving these equations we can get the following expressions:

$$\begin{aligned}
B_1 &= D + \frac{d_1\gamma_1^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2\alpha_2}{(d_2-1)}\gamma_1 - (\beta_2 + rK_2)\right] \\
B_2 &= D + \frac{d_1\gamma_2^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2\alpha_2}{(d_2-1)}\gamma_2 - (\beta_2 + rK_2)\right] \\
&= \frac{d_1\gamma_3^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2\alpha_1}{(d_2-1)}\gamma_3 - (\beta_1 + rK_1)\right] \\
A_1 &= C - \frac{d_2\gamma_2^{-d_1}}{(d_1-d_2)r}\left[\frac{d_1\alpha_2}{(d_1-1)}\gamma_2 - (\beta_2 + rK_2)\right] \\
&= A - \frac{d_2\gamma_3^{-d_1}}{(d_1-d_2)r}\left[\frac{d_1\alpha_1}{(d_1-1)}\gamma_3 - (\beta_1 + rK_1)\right].
\end{aligned}$$

The threshold γ_1 is a solution to the following equation:

$$\phi(\gamma_1, K_2) := C(d_1 - d_2)\gamma_1^{d_1} + (1 - d_2)\frac{\alpha_2}{r-\mu}\gamma_1 + d_2\left(\frac{\beta_2}{r} + K_2\right) = 0$$

and the thresholds γ_2 and γ_3 are a solution to the system of equations

$$\begin{aligned}
D(d_1-d_2)r + d_1\gamma_2^{-d_2}\left[\frac{d_2\alpha_2}{(d_2-1)}\gamma_2 - (\beta_2 + rK_2)\right] - d_1\gamma_3^{-d_2}\left[\frac{d_2\alpha_1}{(d_2-1)}\gamma_3 - (\beta_1 + rK_1)\right] &= 0 \\
(C-A)(d_1-d_2)r - d_2\gamma_2^{-d_1}\left[\frac{d_1\alpha_2}{(d_1-1)}\gamma_2 - (\beta_2 + rK_2)\right] + d_2\gamma_3^{-d_1}\left[\frac{d_1\alpha_1}{(d_1-1)}\gamma_3 - (\beta_1 + rK_1)\right] &= 0.
\end{aligned}$$

A.2.2 $K_2^+ < K_2$ or $K_1 < K_1^-$

Since $z_2 < \gamma_3 < P_{2,1}$, using the smooth-pasting conditions, we get the following expressions

$$\begin{aligned}
B_2\gamma^{d_2} &= A\gamma^{d_1} + \frac{\alpha_1}{r-\mu}\gamma - \frac{\beta_1}{r} - K_1 \\
d_2B_2\gamma^{d_2} &= d_1A\gamma^{d_1} + \frac{\alpha_1}{r-\mu}\gamma
\end{aligned}$$

We conclude, that

$$B_1 = -\frac{d_1\gamma^{-d_2}}{(d_2-d_1)r}\left[\frac{d_2\alpha_1}{(d_2-1)}\gamma - (\beta_1 + rK_1)\right] \tag{63}$$

and γ_3 satisfies the equation

$$(d_2 - d_1)Ap^{d_1} + (d_2 - 1)\frac{\alpha_1}{r-\mu}p - d_2\left(\frac{\beta_1}{r} + K_1\right) = 0. \tag{64}$$

A.2.3 The bounds K_1^+, K_1^-, K_2^+ and K_2^-

Let us assume that the value function is given by s_2. Then, for values of revenue $p \in (\gamma_1, \gamma_2)$ the firm invests in the less risky project (project 2). It is straightforward that the investment is not optimal if $v_2^*(p) - K_2 < 0$. Thus, $\gamma_1 > \hat{p}$, where $v_2^*(\hat{p}) - K_2 = 0$. Furthermore, since $K_2 > -K_x$, it is not optimal to invest in project 2 for values of $p \leq P_{2x}$. On the other hand, since γ_1 triggers the investment in project 2, $\gamma_1 < z_2$, where is defined in Proposition 2. Additionally, since $K_2 + K_{21} > K_1$, it will be never optimal to invest in project 2 for values of revenue larger than P_{21}. Thus $\gamma_1 < P_{21}$.

Let K_2^- and K_2^+ be respectively the upper and lower bounds for K_2. Thus, K_2^+ is obtained as solution to the equation

$$K_2^+ = \max\{K_2 : \phi(\gamma_1, K_2) = 0, \gamma_1 \in [P_{2x}, P_{21}]\}, \tag{65}$$

and K_2^- is obtained as solution to the equation

$$K_2^- = \min\{K_2 : \phi(\gamma_1, K_2) = 0, \gamma_1 \in [P_{2x}, P_{21}]\}, \tag{66}$$

where ϕ is defined in Section A.2.1. These two equations have to be solved in K_2 because all the remaining parameters are fixed. The parameter K_2^- can be explicitly computed because the pair $(\gamma_1, K_2) = (P_{2x}, -K_x)$ solves the equation $\phi(\gamma_1, K_2) = 0$. Since we are imposing that $K_2 > -K_x$, then $K_2^- = -K_x$.

Fix now $K_2 \in (K_2^-, K_2^+)$, then γ_1 can be obtained following Section A.2.1. Considering the structure of s_2, one has that $0 < \gamma_1 < \gamma_2 < \gamma_3$. Thus, following the same line of reasoning, one can obtain K_1^- and K_1^+ fixing $\gamma_2 = \gamma_1$ and $\gamma_2 = \gamma_3$ and solving the system of equations obtained at the end of Section A.2.1, in K_1.

Considering $\gamma_2 = \gamma_3 = \gamma$, we get

$$\begin{aligned}
D(d_1 - d_2)r + d_1\gamma^{-d_2}\left[\frac{d_2\alpha_2}{(d_2-1)}\gamma - (\beta_2 + rK_2)\right] - d_1\gamma^{-d_2}\left[\frac{d_2\alpha_1}{(d_2-1)}\gamma - (\beta_1 + rK_1)\right] &= 0\\
(C - A)(d_1 - d_2)r - d_2\gamma^{-d_1}\left[\frac{d_1\alpha_2}{(d_1-1)}\gamma - (\beta_2 + rK_2)\right] + d_2\gamma^{-d_1}\left[\frac{d_1\alpha_1}{(d_1-1)}\gamma - (\beta_1 + rK_1)\right] &= 0.
\end{aligned}$$

One can easily see that $(\gamma, K_1) = (P_{21}, K_2 + K_{21})$ solves the system of equations. As we are imposing the condition $K_1 < K_2 + K_{21}$, this implies that $K_1^+ = K_2 + K_{21}$.

Fix now $\gamma_2 = \gamma_1$. Then, the lower bound can be found solving the system

$$\begin{aligned}
D(d_1 - d_2)r + d_1\gamma_1^{-d_2}\left[\frac{d_2\alpha_2}{(d_2-1)}\gamma_1 - (\beta_2 + rK_2)\right] - d_1\gamma_3^{-d_2}\left[\frac{d_2\alpha_1}{(d_2-1)}\gamma_3 - (\beta_1 + rK_1)\right] &= 0\\
(C - A)(d_1 - d_2)r - d_2\gamma_1^{-d_1}\left[\frac{d_1\alpha_2}{(d_1-1)}\gamma_1 - (\beta_2 + rK_2)\right] + d_2\gamma_3^{-d_1}\left[\frac{d_1\alpha_1}{(d_1-1)}\gamma_3 - (\beta_1 + rK_1)\right] &= 0,
\end{aligned}$$

in γ_3 and K_1.

A.3 Investment problem 19

The investor's value is given by the solution of the Bellman equation $r\tilde{u}(p) - \mathcal{L}\tilde{u}(p) = 0$. A solution to this equation is $u(p) = \tilde{A}p^{d_1} + \tilde{B}p^{d_2}$. Taking into account that $\lim_{p\to 0} u(p) = \lim_{p\to\infty} u(p) = 0$, then we get the value function

$$\tilde{W}(p) = \begin{cases} \tilde{B}p^{d_2}, & p \leq \tilde{\gamma}_1 \\ f_1(p), & \tilde{\gamma}_1 < p < \tilde{\gamma}_2 \\ \tilde{A}p^{d_1}, & p \geq \tilde{\gamma}_2 \end{cases}.$$

Contrary to what happens in usual real options models, the value function is not C^1, it is only C^0. This is not a problem if W is still a viscosity solution to the previous ordinary differential equation (for more details see Øksendal and Reikvam (1998)). Providing that the parameters and thresholds are as follows, the value function $\tilde{W}$ is a viscosity solution to the ordinary differential equation.

(i) if $\epsilon_1 P_{12}^{d_1} \geq \epsilon_2 P_h^{d_1}$, then

$$\tilde{\gamma}_1 = P_{1x},\ \tilde{\gamma}_2 = P_h,\ \tilde{B} = \epsilon_1 P_{1x}^{-d_2},\ \text{and}\ \tilde{A} = \epsilon_1 P_h^{-d_1};$$

(ii) $\epsilon_1 P_{12}^{d_1} < \epsilon_2 P_{12}^{d_1}$, then

$$\tilde{\gamma}_1 = P_{1x},\ \tilde{\gamma}_2 = P_h,\ \tilde{B} = \epsilon_1 P_{1x}^{-d_2},\ \text{and}\ \tilde{A} = \epsilon_1 P_{12}^{-d_1};$$

More details on how to prove that $\tilde{W}$ is a viscosity solution can be found in Section 4.4 of Oliveira and Perkowski (2020) where the authors prove that the value function, which is not C^1, is a viscosity solution to the respective HJB equation.

B Proofs

B.1 Proof of Proposition 2

We start by deriving $u^* = \max(v_1(p) - K_1, v_2(p) - K_2)$, with v_1 and v_2 defined in Proposition 1.

Let us assume, without loss of generality, that $K_1 = 0$ and $K_2 = 0$. Then, it is straightforward that

$$v_1(p) = v_2(p) \Leftrightarrow p \in (0, P_{2x}) \cup (P_{12}, P_{21}).$$

For $x \in (P_{2x}, P_{1x})$, it is also trivial that $v_2(p) > v_1(p)$. Taking into account that

$$v_2(P_{1x}) > v_1(P_{1x}) = -K_x \quad \text{and} \quad v_2(P_h) > v_1(P_h) = v_2(P_h) - K_{12},$$

and the monotony of v_2 and v_1 we can conclude that $v_2(p) > v_1(p)$ for $p \in (P_{1x}, P_h)$. Finally, for $p \in (P_h, P_{12})$, $v_2(p) > v_1(p) = v_2(p) - K_{12}$.

In fact, due to the continuity of v_1 and v_2, we can conclude that there is a unique point $z_2 \in (P_{12}, P_{21})$ such that $v_1(z_2) = v_2(z_2)$. Additionally, from the convexity of v_2 in (P_{12}, P_{21}) we get that $v_2(p) > v_1(p)$ for $p < z_2$ and $v_2(p) < v_1(p)$ for $p > z_2$.

Given the continuity of $v_1 - K_1$ in K_1, we know that $z_2(K_1)$ is increasing and

$$lim_{K_1 \to K_2 + K_{21}} = P_{21}.$$

Additionally, when we consider $0 < K_1 < K_{12}$ then $v_1(p) - K_1 < v_2(p)$ for $p \in (0, z_2)$ and $v_1(p) - K_1 > v_2(p)$ when $p > z_2$. One can easily check that the analysis made is still true when we start by considering $K_1 = K_2 > 0$.

To finalise this part of the proof, one need to check what happens when we decrease K_1 taking into account that $K_2 < K_1 + K_{12}$.

Let us consider the limit case $K_2 = K_1 + K_{12}$. In this case, $v_2(p) - K_2 < v_1(p) - K_1$ for $p \in (0, P_{2x})$. Additionally, $v_2(p) - K_2 < v_1(p) - K_1$ for $p \in (P_{2x}, z_1)$ with $z_1 \in (P_{2x}, P_h)$, $v_2(p) - K_2 > v_1(p) - K_1$ for $p \in (z_1, P_h)$, $v_2(p) - K_2 = v_1(p) - K_1$ for $p \in (P_h, P_{12})$ and $v_2(p) - K_2 < v_1(p) - K_1$ when $p > P_{12}$. Therefore, in light of the continuity of $v_2 - K_2$ in k_2, , we get the result when we consider $K_1 < K_2 < K_1 + K_{12}$.

B.2 Proof of Proposition 3

Let us assume the following scenario: the set of initial parameters is such that the optimal switching strategy is the "hysteresis strategy" and the value function is in Proposition 1 and there is $z_1 > P_{1x}$ such that the investment threshold γ is such that $\gamma \in (P_{1x}, P_h)$. From standard real options analysis (see for instance Dixit and Pindyck (1994)) we know that the smooth paste condition are given by

$$\begin{cases} B_0\gamma^{d_2} = E\gamma^{d_1} + F\gamma^{d_2} + \frac{\alpha_1}{r-\mu}\gamma - \frac{\beta_1}{r} - K_1 \\ d_2 B_0 \gamma^{d_2-1} = E d_1 \gamma^{d_1-1} + F d_2 \gamma^{d_2-1} + \frac{\alpha_1}{r-\mu} \end{cases}. \tag{67}$$

One may notice that, by definition of the threshold P_{1x}, the pair $(\gamma, B_0) = (P_{1x}, 0)$ is a solution to the system (67) for $K_1 = -K_x$. Given the analysis made for the switching problem, it is known that there is a unique solution $(P_{2x}, P_{1x}, P_h, P_{12}, P_{21})$ such that $P_{2x} < P_{1x} < P_h < P_{12} < P_{21}$. Therefore, fixing E and F as defined (37) and (38), the arguments above allow us to conclude that

$$\begin{cases} 0 = E\gamma^{d_1} + F\gamma^{d_2} + \frac{\alpha_1}{r-\mu}\gamma - \frac{\beta_1}{r} + K_x \\ 0 = E d_1 \gamma^{d_1-1} + F d_2 \gamma^{d_2-1} + \frac{\alpha_1}{r-\mu} \end{cases}.$$

has a unique solution, P_{1x}, for $0 < \gamma < P_h$.

To get our conclusions, we analyse a perturbed version of system (67), considering $K_1 = -K_x + \varepsilon$. Multiplying the first equation of system (67) by d_2 and the second one by γ, the system can be reduced to a single equation

$$m(\gamma) := (d_2 - d_1)E\gamma^{d_1} + (d_2 - 1)\frac{\alpha_1}{r-\mu}\gamma - \frac{\beta_1}{r}d_2 + K_x d_2 - \varepsilon d_2$$

This equations has two solutions. To prove this statement, we may notice that

$$\lim_{\gamma \to 0^+} m(\gamma) = \lim_{\gamma \to +\infty} m(\gamma) = +\infty \quad \text{and} \quad m''(\gamma) = (d_2 - d_1)d_1(d_1 - 1)E\gamma^{d_1-2} > 0.$$

Additionally, choosing $\varepsilon = 0$, we know from Proposition 1 that there is at least one solution to that equation, which is P_{1x}. Additionally, if there is a second one is greater than P_h. This can be proved noticing that

$$\begin{aligned} m(P_h) &= (d_2 - d_1)EP_h^{d_1} + (d_2 - 1)\frac{\alpha_1}{r-\mu}P_h - \frac{\beta_1}{r}d_2 + K_x d_2 - \varepsilon d_2 \\ &= (d_2 - d_1)EP_h^{d_1} + (d_2 - 1)\frac{\alpha_2}{r-\mu}P_h + \left(K_x - \frac{\beta_2}{r} - K_{12}\right)d_2 - \varepsilon d_2 \end{aligned}$$

the second equality following in light of the smooth paste conditions presented in Appendix A.1.2. Given the expression for E presented in Appendix A.1.2, we get the following

$$\begin{aligned} m(P_h) = &-\left(\frac{P_h}{P_{2x}}\right)^{d_1}\left[(d_2 - 1)\frac{\alpha_2}{r-\mu}P_h + \left(K_x - \frac{\beta_2}{r}\right)d_2\right] + (d_2 - 1)\frac{\alpha_2}{r-\mu}P_h \\ &+ \left(K_x - \frac{\beta_2}{r} - K_{12}\right)d_2 - \varepsilon d_2 \\ = &\underbrace{\left(1 - \left(\frac{P_h}{P_{2x}}\right)^{d_1}\right)}_{>0}\underbrace{\left[(d_2 - 1)\frac{\alpha_2}{r-\mu}P_h + \left(K_x - \frac{\beta_2}{r}\right)d_2\right]}_{<0} - (\varepsilon + K_{12})\,d_2 < 0 \end{aligned}$$

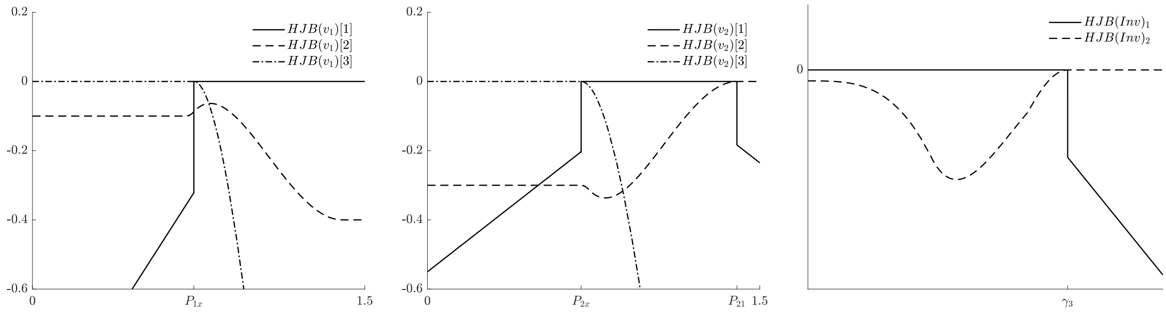
The sign of the second term follows from the fact that $E > 0$ because it is the value of an option.

Due to the continuity of $m(\gamma; \varepsilon) \equiv m(\gamma)$ on ε, for any $\varepsilon > 0$ there are two solutions, one that is smaller than P_{1x} and a second one that is greater than P_h. Both hypothesis contradict the possibility of investment in the hysteresis region.

C Additional figures and tables

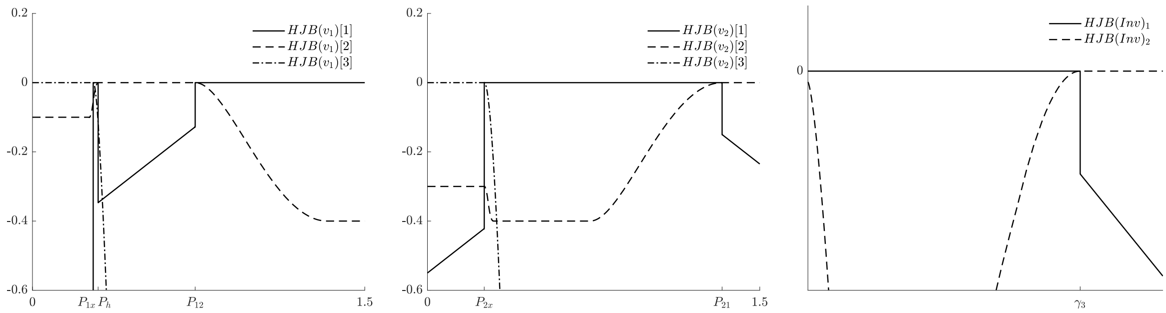
C.1 Numerical Verification of the HJB equations in Section 5.2.1

In Table 9, we illustrate the behaviour of the optimal investment strategy with changing μ. We find that γ_3 is not monotonic with μ. To verify that the behaviour is not a consequence of a numerical error, we present the numerical verification of the HJB equations for the following values of μ: $\mu = -0.030$ and $\mu = 0.025$ (these are the values of μ where the monotony of γ_3 changes). For each value of μ we present three plots since we have to compute the value functions v_1, v_2 and W. As the HJB equations are written as the maximum between three terms for the switching problem and two terms for the investment problem, all these terms must be non-positive and at least one of them must be equal to zero. Figure 15 shows the verification plots for $\mu = -0.03$ and Figure 16 shows the verification plots for $\mu = 0.025$.



(a) (b) (c)

Figure 15: $\mu = -0.030$, HJB verification for v_1 (panel (a)), for v_2 (panel (b)) and for the investment problem (panel (c)).



(a) (b) (c)

Figure 16: Case: $\mu = 0.025$, HJB verification for v_1 (panel (a)), for v_2 (panel (b)) and for the investment problem (panel (c)).

C.2 Switching parameters of Section 5.2.2

In Table 15, we present the switching thresholds regarding the set of parameters used in section 5.2.2. For the reader convenience, we provide here the parameters considered: $\mu = 0$, $r = 0.05$, $\alpha_1 = 1$, $\beta_1 = 1$, $\alpha_2 = 0.6$, $\beta_2 = 0.5$, $K_x = -1$, $K_{12} = 0.25$, $K_{21} = 0.5$.

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σ	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	Strategy
0.1000	0.7198	0.7782	1.0017	0.6536	1.6142	Hyst
0.1500	0.6107	0.6841	0.9400	0.5451	1.7371	Hyst
0.2000	0.5183	0.5999	0.8894	0.4550	1.8566	Hyst
0.2400	0.4556	0.5407	0.8546	0.3947	1.9514	Hyst
0.2500	0.4413	0.5270	0.8465	0.3811	1.9752	Hyst
0.3000	0.3775	0.4649	0.8091	0.3208	2.0942	Hyst

Table 15: Switching thresholds for the illustration in Section 5.2.2.

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