ASYMPTOTIC DYNAMICS OF HAMILTONIAN POLYMATRIX REPLICATORS

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ABSTRACT. In a previous paper [3] we have studied flows on polytopes presenting a method to encapsulate its asymptotic dynamics along the heteroclinic network formed by the polytope's edges and vertices. These results apply to the class of polymatrix replicator systems, which contains several important models in Evolutionary Game Theory. Here we establish the Hamiltonian character of the asymptotic dynamics of Hamiltonian polymatrix replicators.

CONTENTS

1. INTRODUCTION

A new method to study the asymptotic dynamics of flows defined on polytopes is presented in [3]. This method allows us to study the asymptotic dynamics of flows defined on polytopes along the heteroclinic network formed out of the polytope's vertices and edges. Examples of such dynamical systems arise naturally in the context of evolutionary game theory (EGT) developed by J. Maynard Smith and G. R. Price [18].

One such example is the *polymatrix replicator*, introduced in [2, 4], that is a system of ordinary differential equations developed to study the dynamics of the designated *polymatrix game*. This game models the time evolution of the strategies that individuals from a stratified population choose to interact with each other.

The polymatrix replicator induces a flow in a polytope defined by a finite product of simplices. These systems extend the class of the replicator and the bimatrix replicator equations studied in [19] and in [16, 17], respectively.

In [2] the authors have introduced the subclass of conservative polymatrix replicators (see Definition 4.4) which are Hamiltonian systems w.r.t. appropriate Poisson structures. Moreover, in [4] the study of this subclass is developed toghether with the subclass of dissipative polymatrix replicators.

In this paper we will study the asymptotic dynamics of the conservative polymatrix replicators, studying in particular its Hamiltonian character. Namely, the main result of this paper states that for conservative polymatrix replicators a flow map defined in the dual of the phase space is Hamiltonian w.r.t. some appropriate Poisson structure on a system of cross sections (Theorem 6.19).

For this, we need to remember the main strategy of the method developed in [3], trying to avoid as much as possible the formal definitions as well the technical details that can be seen in more detail in the referred paper.

The paper is organized as follows. In Section 2 we establish the necessary background (based on [3]) to contextualize the results of this paper. In particular we outline the construction of the asymptotic dynamics for a large class of flows on polytopes that includes the polymatrix replicators. In Section 3 we define a Poincaré map for Hamiltonian systems on Poisson manifolds. In Section 4 we provide a short introduction to polymatrix replicators, following [2]. Namely, we state the basic definitions and results for the class of conservative polymatrix replicators, that, based on the main results of [2, 4], we designate as *Hamiltonian polymatrix replicators*. In Section 5 we recall the technique developed in [3] to analyze the asymptotic dynamics of a flow along the heteroclinic network formed by the edges and vertices of the polytope where it is defined. In particular we review the main definitions and results for the polymatrix replicator vector field. In Section 6 we study the Poisson geometric properties of the Poincaré maps in the case of Hamiltonian polymatrix replicators. Finally, in Section 7 we present an example of a Hamiltonian polymatrix replicator with a non trivial dimension to provide an illustration of the concepts and main results of this paper. The graphics in this section were produced with *Wolfram Mathematica* and *Geogebra* software.

2. Outline of the construction

We now outline the construction of the asymptotic dynamics for a large class of flows on polytopes that includes the polymatrix replicators. A *polytope* is a compact convex set in some Euclidean space obtained as the intersection of finitely many half-spaces. A polytope is called *simple* if the number of edges (or facets) incident with each vertex equals the polytope's dimension. Prisms, phase spaces of polymatrix replicators, are examples of simple polytopes. In [3] we consider and study analytic vector fields on simple polytopes which have the property of being tangent to every face of the polytope. Such vector fields induce complete flows on the polytope which leave all faces invariant. Vertexes of the polytope are singularities of the vector field, while edges without singularities, called *flowing edges*, consist of single orbits flowing between two end-point vertexes. The vertexes and flowing edges form a heteroclinic network of the vector field. The purpose of this construction is to analyse the asymptotic dynamics of the vector field along this one-dimensional skeleton. Throughout the text we assume that every vector field is *regular*. This means that the transversal derivative of the vector field is never identically zero along any facet of the polytope.

The analysis of the vector field's dynamics along its edge heteroclinic network makes use of Poincaré maps between cross sections tranversal to the flowing edges. Any Poincaré map along a heteroclinic or homoclinic orbit is a composition of two types of maps, global and local Poincaré maps. A *global map*, denoted by P_γ, is defined in a tubular neighbourhood of any flowing-edge γ. It maps points between two cross sections Σ_γ^- and Σ_γ^+ transversal to the flow along the edge γ. A *local map*, denoted by P_v, is defined in a neighbourhood of any vertex singularity v. For any pair of flowing-edges γ, γ' such that v is both the ending point of γ' and the starting point of γ, the local map P_v takes points from $\Sigma_{\gamma'}^+$ to Σ_γ^-. See Figure 1.

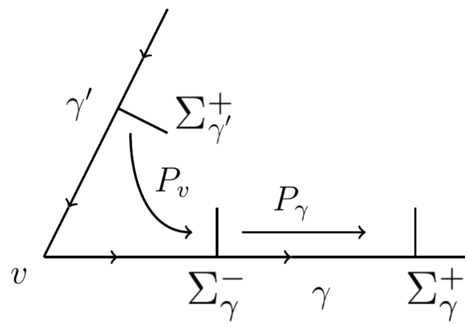

FIGURE 1. Local and global Poincaré maps along a heteroclinic orbit.

Asymptotically, the nonlinear character of the global Poincaré maps fade away as we approach a heteroclinic orbit. This means that these non-linearities are irrelevant for the asymptotic analysis. For regular vector fields, the *skeleton character* at a vertex, defined as the set of eigenvalues of the tangent map along the edge eigen-directions, completely determines the asymptotic behaviour of the local Poincaré map at that vertex.

To describe the limit dynamical behaviour we introduce the *dual cone* of a polytope where the asymptotic piecewise linear dynamics unfolds. This space lies inside $\mathbb{R}^F$, where F is the set of the polytope's facets. The dual cone of a d-dimensional simple polytope Γ is the union

$$\mathcal{C}^*(\Gamma) := \bigcup_{v \in V} \Pi_v \ ,$$

where for each vertex v, Π_v is the d-dimensional sector consisting of points $y \in \mathbb{R}^F$ with non-negative coordinates such that $y_\sigma = 0$ for every facet that does not contain v. See Figure 2.

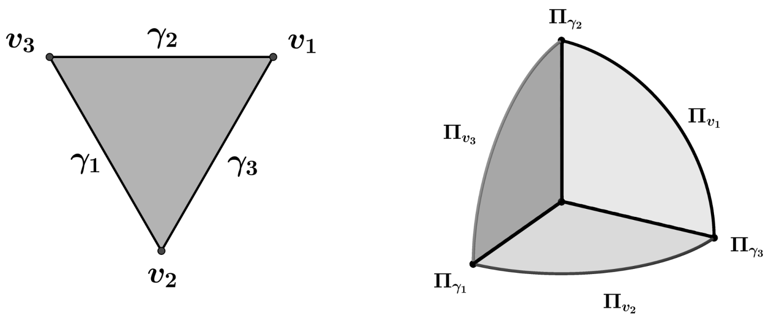

FIGURE 2. Dual cone of a triangle in $\mathbb{R}^F$.

Given a vector field X on a d dimensional polytope $\Gamma \subset \mathbb{R}^d$, we now describe a rescaling change of coordinates Ψ_ϵ^X, depending on a blow up parameter ϵ. See Figure 3.

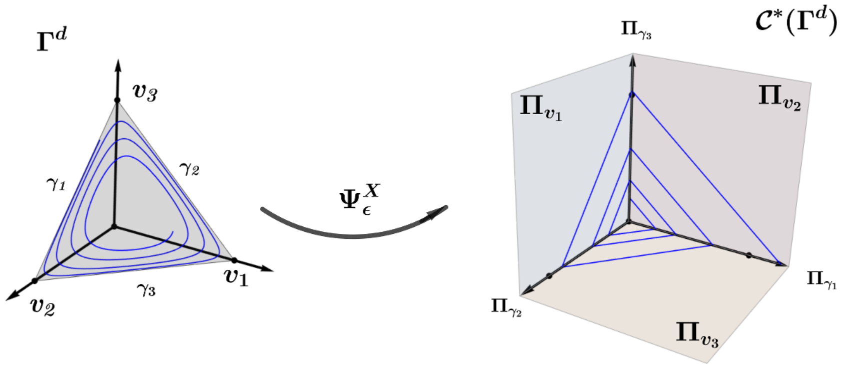

FIGURE 3. Asymptotic linearisation on the dual cone. The left image represents an orbit on the simplex Δ^2 and the right one the corresponding (nearly) piecewise linear image under the map Ψ_ϵ^X on the dual cone.

This change of coordinates maps tubular neighbourhoods of edges and vertices to the dual cone $\mathcal{C}^*(\Gamma)$. For instance, the tubular neighbourhood N_v of a vertex v is defined as follows. Consider a system

$(x_1, \ldots, x_d)$ of affine coordinates around v, which assigns coordinates $(0, \ldots, 0)$ to v and such that the hyperplanes $x_j = 0$ are precisely the facets of the polytope through v. Then N_v is defined by

$$N_v := \{p \in \Gamma^d \colon 0 \leq x_j(p) \leq 1 \text{ for } 1 \leq j \leq d\}.$$

The sets $\{x_j = 0\} \cap N_v$ are called the outer facets of N_v. The remaining facets of N_v, defined by equations like $x_i = 1$, are called the inner facets of N_v. The previous cross sections $\Sigma_\gamma^\pm$ can be chosen to match these inner facets of the neighbourhoods N_v.

The rescaling change of coordinates Ψ_ϵ^X maps N_v to the sector Π_v. Enumerating F so that the facets through v are precisely $\sigma_1, \ldots, \sigma_d$, the map Ψ_ϵ^X is defined on the neighbourhood N_v by

$$\Psi_\epsilon^X(q) := (-\epsilon^2 \log x_1(q), \ldots, -\epsilon^2 \log x_d(q), 0, \ldots, 0).$$

Similarly, given an edge γ, Ψ_ϵ^X maps a tubular neighbourhood N_γ of γ to the facet sector $\Pi_\gamma := \Pi_v \cap \Pi_{v'}$ of Π_v where v' is the other end-point of γ. The map Ψ_ϵ^X sends interior facets of N_v and N_γ respectively to boundary facets of Π_v and Π_γ while it maps outer facets of N_v and N_γ to infinity. As the rescaling parameter ϵ tends to 0, the rescaled push-forward $\epsilon^{-2}\,(\Psi_\epsilon^X)_* X$ of the vector field X converges to a constant vector field χ^v on each sector Π_v. This means that asymptotically, as $\epsilon \to 0$, trajectories become lines in the coordinates $(y_\sigma)_{\sigma \in F} = \Psi_\epsilon^X$. Given a flowing-edge γ between vertices v and v', the map Ψ_ϵ^X over N_γ depends only on the coordinates transversal to γ. Moreover, as $\epsilon \to 0$ the global Poincaré map P_γ converges to the identity map in the coordinates $(y_\sigma)_{\sigma \in F} = \Psi_\epsilon^X$. Hence the sector Π_γ is naturally identified as the common facet between the sectors Π_v and $\Pi_{v'}$. Hence the asymptotic dynamics along the vertex-edge heteroclinic network is completely determined by the vector field's geometry at the vertex singularities and can be described by a piecewise constant vector field χ on the dual cone, whose components are precisely those of the skeleton character of X. We refer to this piecewise constant vector field as the *skeleton vector field* of X. This vector field χ induces a piecewise linear flow on the dual cone whose dynamics can be computationally explored.

We use Poincaré maps for a global analysis of the asymptotic dynamics of the flow of X. We consider a subset S of flowing-edges with the property that every heteroclinic cycle goes through at least one edge in S. Such sets are called structural sets. The flow of X induces a Poincaré map P_S on the system of cross sections $\Sigma_S := \cup_{\gamma \in S} \Sigma_\gamma^+$. Each branch of the Poincaré map P_S is associated with a heteroclinic path starting with an edge in S and ending at its first return to another edge in S. These heteroclinic paths are the branches of S. The flow of the skeleton vector field χ also induces a first return map $\pi_S : D_S \subset \Pi_S \to \Pi_S$ on the system of cross sections $\Pi_S := \cup_{\gamma \in S} \Pi_\gamma$. This map π_S, called the *skeleton flow map*, is piecewise linear and its domain is a finite union of open convex cones. In some cases, see Proposition 5.18, the map π_S becomes a closed dynamical system.

We can now recall the main result in [3], Theorem 5.20 below, which says that in the rescaling change of coordinates Ψ_ϵ^X, the Poincaré map P_S converges in the C^∞ topology to the skeleton flow map π_S, in the sense that the following limit holds

$$\lim_{\epsilon\to 0} \Psi_\epsilon^X \circ P_S \circ (\Psi_\epsilon^X)^{-1} = \pi_S$$

with uniform convergence of the map and its derivatives over any compact set contained in the domain $D_S \subset \Pi_S$.

Consider now, for each facet σ of the polytope, an affine function $\mathbb{R}^d \ni q \mapsto x_\sigma(q) \in \mathbb{R}$ which vanishes on σ and is strictly positive on the rest of the polytope. With this family of affine functions we can present the polytope as $\Gamma^d = \cap_{\sigma\in F}\{x_\sigma \geq 0\}$. Any function function $h : \mathrm{int}(\Gamma^d) \to \mathbb{R}$ of the form

$$h(q) = \sum_{\sigma\in F} c_\sigma \, \log x_\sigma(q) \quad (c_\sigma \in \mathbb{R})$$

rescales to the following piecewise linear function on the dual cone

$$\eta(y) := \sum_{\sigma\in F} c_\sigma \, y_\sigma$$

in the sense that $\eta = \lim_{\epsilon\to 0} \epsilon^{-2}\,(h \circ (\Psi_\epsilon^X)^{-1})$. When all coefficients c_σ have the same sign then η is a proper function on the dual cone and all levels of η are compact sets. If the function h is invariant under the flow of X, i.e. $h \circ P_S = h$, then the piecewise linear function η is also invariant under the skeleton flow, i.e. $\eta \circ \pi_S = \eta$. Thus integrals of motion (of vector fields on polytopes) of the previous form give rise to (asymptotic) piecewise linear integrals of motion for the skeleton flow.

3. Poisson Poincaré maps

In this section we will define Poincaré map for Hamiltonian systems on Poisson manifolds. For Hamiltonian vector fields on symplectic manifolds it is well known that the Poincaré map preserves the induced symplectic structure on any transversal section (see [20, Theorem 1.8.]). We extend this fact to Hamiltonian systems on Poisson manifolds, showing that any transversal section inherits a Poisson structure and the Poincaré map preserves this structure.

A Poisson manifold is a pair (M, π) where M is a smooth manifold without boundary and π a Poisson structure on M. Recall that a Poisson structure is a smooth bivector field π with the property that $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket (cf. e.g. [10]). The bivector field π defines a vector bundle map

$$\pi^\sharp \colon T^*M \to TM \quad \text{by} \quad \xi \to \pi(\xi, .). \tag{3.1}$$

The image of this map is an integrable singular distribution which integrates to a symplectic foliation, i.e., a foliation whose leaves have a symplectic structure induced by the Poisson structure.

Notice that a Poisson structure can also be defined as a Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)\times C^\infty(M)$ satisfying the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \qquad f, g, h \in C^\infty(M).$$

These two descriptions are related by $\pi(\mathrm{d}f, \mathrm{d}g) = \{f, g\}$. In a local coordinate chart $(U, x_1, .., x_n)$, or equivalently when $M = \mathbb{R}^n$, a Poisson bracket takes the form

$$\{f, g\}(x) = (\mathrm{d}_x f)^t \left[\, \pi_{ij}(x)\right]_{ij} \mathrm{d}_x g,$$

where $\pi(x) = \left[\, \pi_{ij}(x)\right]_{ij} = \left[\, \{x_i, x_j\}(x)\right]_{ij}$ is a skew symmetric matrix valued smooth function satisfying

$$\sum_{l=1}^{n} \frac{\partial \pi_{ij}}{\partial x_l}\pi_{lk} + \frac{\partial \pi_{jk}}{\partial x_l}\pi_{li} + \frac{\partial \pi_{ki}}{\partial x_l}\pi_{lj} = 0 \qquad \forall i, j, k\ .$$

Definition 3.1. Let $(M, \{,\}_M)$ and $(N, \{.,.\}_N)$ be two Poisson manifolds. A smooth map $\psi : M \to N$ will be called a *Poisson map* iff

$$\{f\circ\psi, h\circ\psi\}_M = \{f, h\}_N \circ \psi \quad \forall f, h \in C^\infty(N). \tag{3.2}$$

Using the map $\pi^\sharp$, defined at (3.1), this condition reads as

$$(D\psi)\pi^\sharp_M (D\psi)^* = \pi^\sharp_N \circ \psi, \tag{3.3}$$

where we use the notation $(D\psi)^*$ to denote the adjoint operator of $D\psi$. Notice that, if $D\psi$ is the Jacobian matrix of ψ in local coordinates, then the matrix representation of the pullback will be $(D\psi)^t$.

Remark 3.2. When ψ is a diffeomorphism and only one of the manifolds M or N is Poisson manifold, Definition 3.1 can be used to push-forward or pullback the Poisson structure to the other manifold.

Definition 3.3. Let (M, π) be a Poisson manifold. The Hamiltonian vector field associated to a given function $H : M \to \mathbb{R}$ is defined by derivation $X_H(f) := \{H, f\}$ for $f \in C^\infty(M)$, or equivalently $X_H := \pi^\sharp(dH)$.

As in the symplectic case, to define the Poincaré map we will consider the traversal sections inside the level set of the Hamiltonian. We will show that such a transversal section is a cosymplectic submanifolds of the ambient Poisson manifold and naturally inherits a Poisson structure. For more details on cosymplectic submanifolds see [21, Section 5.1].

Definition 3.4. $N \subset (M, \pi)$ is a *cosymplectic submanifold* if it is the level set of *second class constraints* i.e., $N = \cap_{i=1}^{2k} G_i^{-1}(0)$ where $\{G_1, ..., G_{2k}\}$ are functions such that $[\{G_i, G_j\}(x)]_{i,j}$ is an invertible matrix at all points $x \in N$.

Remark 3.5. A constraint is called first class if it Poisson commutes with other constraints of the system. Sometimes, in the literature, a constraint that has non-zero Poisson bracket with at least one other constraint of the system is called a second class constraint. Definition 3.4 demands a stronger condition, but the cosymplectic submanifolds that we will use have codimension 2, where having non-zero Poisson bracket with the other constraint is the same as $[\{G_i, G_j\}(x)]_{i,j=1,2}$ being an invertible matrix.

Every, cosympletic submanifold is naturally equipped with a Poisson bracket called Dirac bracket. Paul Dirac, [7], developed this bracket to treat classical systems with second class constraints in Hamiltonian mechanics.

Definition 3.6. For cosymplectic submanifold $N \subset (M, \pi)$, let

$$G_1, .., G_{2k} : U \to \mathbb{R}$$

be its second class constraints, where U is a small enough neighbourhood of N in M such that the matrix $[\{G_i, G_j\}(x)]_{i,j}$ is invertible at all points $x \in U$. The *Dirac bracket* is defined on $\mathrm{C}^\infty(U)$ by

$$\{f, g\}_{\text{Dirac}} = \{f, g\} - [\{f, G_i\}]^t \, [\{G_i, G_j\}]^{-1} \, [\{G_i, g\}], \tag{3.4}$$

where $[\{., G_i\}]$ is the column matrix with components $\{., G_i\}$ $i = 1, \ldots, 2k$.

Dirac bracket is actually a Poisson bracket on the open submanifold U, see [21]. It takes an easy calculation to see that constraint functions G_i, $i = 1, \ldots, 2k$ are Casimirs of Dirac bracket. This fact allows the restriction of Dirac bracket to the cosymplectic submanifold N. Note that, in general, restricting (pulling back) a Poisson structure to an arbitrary submanifold is not straightforward. Actually, the decomposition

$$\pi^\sharp(T_xN^\circ) \oplus T_xN = T_xM \tag{3.5}$$

that holds for every point $x \in N$ and a strait forward calculation yield the independence of the extension in the following definition. In Equation (3.5), the term T_xN° is the annihilator of T_xN in T_x^*M. We will use this notation in the rest of the paper. Equation (3.5) can be used as Definition of a cosymplectic submanifold, see [21], but for our propose it suits better to use of second class constraints to define cosymplectic submanifolds.

Definition 3.7. The restricted Dirac bracket on cosymplectic submanifold N, which will be also referred to as *Dirac bracket,* is simply defined by extending *in any arbitrary way* functions on N to functions on U, calculating their Dirac bracket on U and restricting the result back to N.

We consider a Hamiltonian H on the m-dimensional Poisson manifold (M, π) and its associated Hamiltonian vector field defined by

$X_H = \{H, .\} = \pi^\sharp(dH)$. For a given point $x_0 \in M$ let U be a neighbourhood around it such that $X_H(x) \neq 0 \quad \forall x \in U$, and $\mathcal{E}_{x_0}$ be the energy surface passing through x_0, i.e., the connected component of $H^{-1}(H(x_0))$ containing x_0. We call *level transversal section* to X_H at a regular point $x_0 \in M$ any $(m-2)$-dimensional transversal section $\Sigma \subset \mathcal{E}_{x_0} \cap U$ through x_0.

The following lemma shows that Σ is a cosymplectic submanifold.

Lemma 3.8. *Every level transversal section Σ is a cosymplectic submanifold of M.*

Proof. Since $d_{x_0}H \neq 0$, there exist a function G locally defined in U (shrink U if necessary) and linearly independent from H such that

$$\Sigma = \mathcal{E}_{x_0} \cap U \cap G^{-1}(G(x_0)).$$

Then, we have

$$\pi(\mathrm{d}H, \mathrm{d}G) = X_H(G) = dG(X_H) \neq 0$$

by transversality. This finishes the proof. □

Remark 3.9. The second term in the right hand side of Equation (3.4) is

$$\begin{bmatrix} \{f, H\} & \{f, G\} \end{bmatrix} \begin{bmatrix} 0 & \{H, G\} \\ \{G, H\} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \{H, g\} \\ \{G, g\} \end{bmatrix}.$$

Then, using extensions $\tilde{f}$ and $\tilde{g}$ of $f, g \in C^\infty(\Sigma)$ such that at every point $x \in \Sigma$ their differentials vanishes on X_H, yields

$$\{f, g\}_{\text{Dirac}} = \{\tilde{f}, \tilde{g}\}|_\Sigma.$$

We will use this fact to simplify our proofs but arbitrary extensions are more suitable for calculating the Dirac structure. This also means that the Poisson structure on the level transversal section Σ is independent of the choice we make for the second class constraint G.

We will use the same notation f for arbitrary extension and reserve the notation $\tilde{f}$ for extension that their differentials vanishes on X_H at every point $x \in \Sigma$. To avoid any possible confusion, we observe that in [6, Section 8] and [6, Section 8] the notation $\tilde{f}$ is used in a slightly different sense.

Remark 3.10. Cosymplectic submanifolds are special examples of the so called Poisson-Dirac submanifolds, see [6, Section 8]. The induced Poisson structure on a Poisson-Dirac submanifold is defined by using extensions such that their differentials vanish on $\pi^\sharp(T\Sigma^\circ)$. In [6, Section 8] and [21, Lemma 5.1] the notation $\tilde{f}$ is used for this type of extensions. For a cosymplectic submanifold Σ given by second class constraints $G_1, ..., G_{2k}$, we have

$$\pi^\sharp(T\Sigma^\circ) = \oplus_{i=1}^{2k} \mathbb{R} X_{G_i}, \tag{3.6}$$

and the Dirac bracket coincides with the bracket induced in this way, see [21, Section 5.1]. In our case, we only have two constraints H, G and requiring the vanishing of the differential only on X_H (or X_G) at every point $x \in \Sigma$ is enough to obtain the same induced Poisson bracket.

For a fixed time t_0, let $x_1 = \phi_H(t_0, x_0)$, where ϕ_H is the flow of the Hamiltonian vector field X_H, and Σ_0, Σ_1 be level transversal sections at x_0 and x_1, respectively. As usual, a Poincaré map $P = \phi_H(\tau(x), x)$ can be defined from an appropriate neighbourhood of x_0 in Σ_0 to a neighborhood of x_1 in Σ_1. The existence of the smooth function $\tau(x)$ is guaranteed by the Implicit Function Theorem. We replace Σ_0 and Σ_1 by the domain and the image of the Poincaré map P.

By Lemma 3.8 both Σ_i, $i = 0, 1$, are cosymplectic submanifolds equipped with Dirac brackets $\{.,.\}_{\mathrm{Dirac}_i}$, $i = 0, 1$. We will show that the Poincaré map P is a Poisson map (see Definition 3.1).

Proposition 3.11. *The Poincaré map*

$$P : (\Sigma_0, \{.,.\}_{\mathrm{Dirac}_0}) \to (\Sigma_1, \{.,.\}_{\mathrm{Dirac}_1})$$

is a Poisson map.

Proof. We define $\tilde{P} : U_0 \to U_1$ by

$$\tilde{P}(x) := \phi_H(\tilde{\tau}(x), x),$$

where $\tilde{\tau}$ is an extension of τ to a neighborhood U_0 of x_0 such that its differential, $d\tilde{\tau}$, vanishes on X_H. Both neighborhood U_0 and U_1 can be shrunk, if necessary, in a way that both Dirac brackets around Σ_0 and Σ_1 are defined in U_0 and U_1, respectively. A straightforward calculation shows that for every point x in the domain of $\tilde{\tau}$, we have

$$D_x\tilde{P} = D_x\phi_H^{\tilde{\tau}(x)} + (d_x\tilde{\tau})X_H(\phi_H^{\tilde{\tau}(x)}(x)),$$

where $\phi_H^{\tilde{\tau}(x)}(.) = \phi_H(\tilde{\tau}(x), .)$. Furthermore, for every $x \in \Sigma_0$, we have

$$\begin{aligned} D_x\tilde{P}(X_H(x)) &= D_x\phi_H^{\tilde{\tau}(x)}(X_H(x)) + (\underbrace{d_x\tilde{\tau}(X_H(x))}_{=0})X_H(\phi_H^{\tilde{\tau}(x)}(x)) \\ &= D_x\phi_H^{\tilde{\tau}(x)}(X_H(x)) = X_H(\phi_H^{\tilde{\tau}(x)}(x)) = X_H(\tilde{P}(x)). \end{aligned} \tag{3.7}$$

Note that $D_x\phi_H^{\tilde{\tau}(x)}$ in Equation (3.7) is the derivative of time-$\tilde{\tau}(x)$ flow of X_H and the fixed time flow maps of X_H are Poisson maps, i.e. it sends Hamiltonian vector fields to Hamiltonian vector field. Furthermore, the flow of X_H preserves H, this means that

$$H(\tilde{P}(x)) = H(\phi_H^{\tilde{\tau}(x)}(x)) = H(x).$$

As we set in Remark 3.9, let $\tilde{f}$ be an extension of a given $f \in C^\infty(\Sigma_1)$ such that

$$d_x\tilde{f}(X_H) = 0, \quad \forall x \in \Sigma_1,$$

then for every $x \in \Sigma_0$,

$$d_x(\tilde{f} \circ \tilde{P})(X_H) = d_{\tilde{P}(x)}\tilde{f} \circ D_x\tilde{P}(X_H) = d_{\tilde{P}(x)}\tilde{f}(X_H) = 0.$$

Now, for $f, g \in C^\infty(\Sigma_1)$ and $x \in \Sigma_0$ we have

$$\begin{aligned}
&\{\tilde{f} \circ \tilde{P}, \tilde{g} \circ \tilde{P}\}(x) = \pi_x\left((D_x\tilde{P})^* \mathrm{d}_x\tilde{f}, (D_x\tilde{P})^* \mathrm{d}_x\tilde{g},\right)\\
&= \pi_x\left((D_x\phi_H^{\bar{\tau}(x)})^* \mathrm{d}_x\tilde{f} + (d_x\bar{\tau}X_H)^* \mathrm{d}_x\tilde{f}, (D_x\phi_H^{\bar{\tau}(x)})^* \mathrm{d}_x\tilde{g} + (d_x\bar{\tau}X_H)^* \mathrm{d}_x\tilde{g}\right)\\
&= \pi_x\left((D_x\phi_H^{\bar{\tau}(x)})^* \mathrm{d}_x\tilde{f}, (D_x\phi_H^{\bar{\tau}(x)})^* \mathrm{d}_x\tilde{g}\right)\\
&= \pi(\mathrm{d}\tilde{f}, \mathrm{d}\tilde{g})(\tilde{P}(x)) = \{\tilde{f}, \tilde{g}\}(\tilde{P}(x)),
\end{aligned}$$

and consequently,

$$\begin{aligned}
\{f \circ P, g \circ P\}_{\mathrm{Dirac}_0} &= \{\tilde{f} \circ \tilde{P}, \tilde{g} \circ \tilde{P}\}|_{\Sigma_0}\\
&= \{\tilde{f}, \tilde{g}\} \circ \tilde{P}|_{\Sigma_0}\\
&= \{f, g\}_{\mathrm{Dirac}_1} \circ P,
\end{aligned}$$

where we used Remark 3.9. This finishes the proof. □

4. Hamiltonian polymatrix replicators

In this section we provide a short introduction to polymatrix replicators, following [2]. In particular we will focus on the class of conservative polymatrix replicators that we designate as *Hamiltonian polymatrix replicators*.

Consider a population divided in p groups where each group is labelled by an integer $\alpha \in \{1, \ldots, p\}$, and the individuals of each group α have exactly n_α strategies to interact with other members of the population (including of the same group). In total we have $n = \sum_{\alpha=1}^{p} n_\alpha$ strategies that we label by the integers $i \in \{1, \ldots, n\}$, denoting by

$$[\alpha] := \{n_1 + \cdots + n_{\alpha-1} + 1, \, \ldots, \, n_1 + \cdots + n_\alpha\} \subset \mathbb{N}$$

the set (interval) of strategies of group α.

Given $\alpha, \beta \in \{1, \ldots, p\}$, consider a real $n_\alpha \times n_\beta$ matrix, say $A^{\alpha,\beta}$, whose entries $a_{ij}^{\alpha,\beta}$, with $i \in [\alpha]$ and $j \in [\beta]$, represent the the average payoff of an individual of the group α using the i^{th} strategy when interacting with an individual of the group β using the j^{th} strategy. Thus the matrix A with entries $a_{ij}^{\alpha,\beta}$, where $\alpha, \beta \in \{1, \ldots, p\}$, $i \in [\alpha]$ and $j \in [\beta]$, is a square matrix of order $n = n_1 + \ldots + n_p$, consisting of the block matrices $A^{\alpha,\beta}$.

Let $\underline{n} = (n_1, \ldots, n_p)$. The *state* of the population is described by a point $x = (x^\alpha)_{1 \le \alpha \le p}$ in the *polytope*

$$\Gamma_{\underline{n}} := \Delta^{n_1 - 1} \times \ldots \times \Delta^{n_p - 1} \subset \mathbb{R}^n \ ,$$

where $\Delta^{n_\alpha-1} = \{x \in \mathbb{R}_+^{[\alpha]} : \sum_{i\in[\alpha]} x_i^\alpha = 1\}$, $x^\alpha = (x_i^\alpha)_{i\in[\alpha]}$ and the entry x_i^α represents the usage frequency of the i^{th} strategy within the group α. We denote by $\partial\Gamma_{\underline{n}}$ the boundary of $\Gamma_{\underline{n}}$.

Assuming random encounters between individuals, for each group $\alpha \in \{1,\dots,p\}$, the average payoff of a strategy $i \in [\alpha]$ within a population with state x is given by

$$(Ax)_i = \sum_{\beta=1}^{p} \left(A^{\alpha,\beta}\right)_i x^\beta = \sum_{\beta=1}^{p} \sum_{k\in[\beta]} a_{ik}^{\alpha,\beta} x_k^\beta \,,$$

where the overall average payoff of group α is given by

$$\sum_{i\in[\alpha]} x_i^\alpha \, (Ax)_i \ .$$

Demanding that the logarithmic growth rate of the frequency of each strategy $i \in [\alpha]$, $\alpha \in \{1,\dots,p\}$, is equal to the payoff difference between strategy i and the overall average payoff of group α yields the system of ordinary differential equations defined on the polytope $\Gamma_{\underline{n}}$,

$$\frac{dx_i^\alpha}{dt} = x_i^\alpha \left((Ax)_i - \sum_{i\in[\alpha]} x_i^\alpha \, (Ax)_i \right), \ \alpha \in \{1,\dots,p\}, i \in [\alpha], \tag{4.1}$$

that will be designated as a *polymatrix replicator*.

The flow $\phi_{\underline{n},A}^t$ of this equation leaves the polytope $\Gamma_{\underline{n}}$ invariant. The argument is analogous to that for the bimatrix replicator equation, see [13, Section 10.3]. Hence, by compactness of $\Gamma_{\underline{n}}$, the flow $\phi_{\underline{n},A}^t$ is complete. From now on we will call 'polymatrix replicator' to system (4.1), to the flow $\phi_{\underline{n},A}^t$ and also to the underlying vector field on $\Gamma_{\underline{n}}$ which we denote by $X_{\underline{n},A}$.

If $p = 1$ equation (4.1) becomes the usual replicator equation with payoff matrix A. When $p = 2$ and $A^{11} = A^{22} = 0$ are null matrices, equation (4.1) becomes the bimatrix replicator equation with payoff matrices A^{12} and $(A^{21})^t$.

Given $\underline{n} = (n_1,\dots,n_p)$, let

$$\mathscr{I}_{\underline{n}} := \{\, I \subset \{1,\dots,n\} \ : \ \#\,(I \cap [\alpha]) \geq 1, \ \forall\, \alpha = 1,\dots,p \,\} \ .$$

A set $I \in \mathscr{I}_{\underline{n}}$ determines the face $\sigma_I := \{\, x \in \Gamma_{\underline{n}} \ : \ x_j = 0, \forall\, j \notin I \,\}$ of $\Gamma_{\underline{n}}$. The correspondence between labels in $\mathscr{I}_{\underline{n}}$ and faces of $\Gamma_{\underline{n}}$ is bijective.

Remark 4.1. The partition of $\Gamma_{\underline{n}}$ into the interiors $\sigma_I^\circ := \operatorname{int}(\sigma_I)$, with $I \in \mathscr{I}_{\underline{n}}$, is a smooth stratification of $\Gamma_{\underline{n}}$ with strata σ_I°. Every stratum σ_I° is a connected open submanifold and for any pair $\sigma_{I_1}^\circ$, $\sigma_{I_2}^\circ$ if $\sigma_{I_1}^\circ \cap \sigma_{I_2} \neq \emptyset$ then $\sigma_{I_1} \subset \sigma_{I_2}$. For more on smooth stratification see [12] and references therein.

For a set $I \in \mathscr{I}_{\underline{n}}$ consider the pair $(\underline{n}^I, A_I)$, where $\underline{n}^I = (n_1^I,\dots,n_p^I)$ with $n_\alpha^I = \#(I \cap [\alpha])$, and $A_I = [\, a_{ij}]_{i,j\in I}$.

Proposition 4.2. [2, Proposition 3] *Given $I \in \mathscr{I}_{\underline{n}}$, the face σ_I of $\Gamma_{\underline{n}}$ is invariant under the flow of $X_{\underline{n},A}$ and the restriction of* (4.1) *to σ_I is the polymatrix replicator $X_{\underline{n}^I,A_I}$.*

For a fixed $\underline{n} = (n_1, \ldots, n_p)$ the correspondence $A \mapsto X_{\underline{n},A}$ is linear and its kernel consists of the matrices $C = \left(C^{\alpha,\beta}\right)_{1\leq\alpha,\beta\leq p}$ where each block $C^{\alpha,\beta}$ has equal rows, i.e., has the form

$$C^{\alpha,\beta} = \begin{pmatrix} c_1^{\alpha,\beta} & c_2^{\alpha,\beta} & \ldots & c_n^{\alpha,\beta} \\ c_1^{\alpha,\beta} & c_2^{\alpha,\beta} & \ldots & c_n^{\alpha,\beta} \\ \vdots & \vdots & & \vdots \\ c_1^{\alpha,\beta} & c_2^{\alpha,\beta} & \ldots & c_n^{\alpha,\beta} \end{pmatrix}.$$

Thus $X_{\underline{n},A_1} = X_{\underline{n},A_2}$ if and only if for every $\alpha, \beta \in \{1, \ldots, p\}$ the matrix $A_1^{\alpha,\beta} - A_2^{\alpha,\beta}$ has equal rows (see [2, Proposition 1]).

We have now the following characterization of the interior equilibria.

Proposition 4.3. [2, Proposition 2] *Given a polymatrix replicator $X_{\underline{n},A}$, a point $q \in \mathrm{int}(\Gamma_{\underline{n}})$ is an equilibrium of $X_{\underline{n},A}$ iff $(A\,q)_i = (A\,q)_j$ for all $i, j \in [\alpha]$ and $\alpha = 1, \ldots, p$.*

In particular the set of interior equilibria of $X_{\underline{n},A}$ is the intersection of some affine subspace with $\mathrm{int}(\Gamma_{\underline{n}})$.

Definition 4.4. A polymatrix replicator $X_{\underline{n},A}$ is said to be *conservative* if there exists:

(a) a point $q \in \mathbb{R}^n$, called *formal equilibrium*, such that $(A\,q)_i = (A\,q)_j$ for all $i, j \in [\alpha]$, and all $\alpha = 1, \ldots, p$ and $\sum_{j\in[\alpha]} q_j = 1$;

(b) matrices $A_0, D \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ such that
   (i) $X_{\underline{n},A_0 D} = X_{\underline{n},A}$,
   (ii) A_0 is a skew symmetric, and
   (iii) $D = \mathrm{diag}(\lambda_1 I_{n_1}, \ldots, \lambda_p I_{n_p})$ with $\lambda_\alpha \neq 0$ for all $\alpha \in \{1, \ldots, p\}$.

The matrix A_0 will be referred to as a *skew symmetric model* for $X_{\underline{n},A}$, and $(\lambda_1, \ldots, \lambda_p) \in (\mathbb{R}^*)^p$ as a *scaling co-vector*.

In [4], another characterization of conservative polymatrix replicators, using quadratic forms, is provided. Furthermore, in [1] the concept of conservative replicator equations (where $p = 1$) is generalized using Dirac structures.

In what follows, the vectors in $\mathbb{R}^n$, or $\mathbb{R}^{[\alpha]}$, are identified with column vectors. Let $\mathbb{1}_n = (1, .., 1)^t \in \mathbb{R}^n$. We will omit the subscript n whenever the dimension of this vector is clear from the context. Similarly, we write $I = I_n$ for the $n \times n$ identity matrix. Given $x \in \mathbb{R}^n$, we denote by D_x the $n \times n$ diagonal matrix $D_x := \mathrm{diag}(x_1, \ldots, x_n)$. For each $\alpha \in \{1, \ldots, p\}$ we define the $n_\alpha \times n_\alpha$ matrix

$$T_x^\alpha := x^\alpha\, \mathbb{1}^t - I\ ,$$

and T_x the $n \times n$ block diagonal matrix $T_x := \mathrm{diag}(T_x^1, \ldots, T_x^p)$.

Given an anti-symmetric matrix A_0, we define the skew symmetric matrix valued mapping $\pi_{A_0} : \mathbb{R}^n \to \mathrm{Mat}_{n\times n}(\mathbb{R})$

$$\pi_{A_0}(x) := (-1)\, T_x\, D_x\, A_0\, D_x\, T_x^t. \tag{4.2}$$

The interior of the polytope $\Gamma_{\underline{n}}$, denoted by $\mathrm{int}(\Gamma_{\underline{n}})$, equipped with π_{A_0} is a Poisson manifold, see [2, Theorem 3.5]. Furthermore,

Theorem 4.5. [2, Theorem 3.7] *Consider a conservative polymatrix replicator $X_{\underline{n},A}$ with formal equilibrium q, skew symmetric model A_0 and scaling co-vector $(\lambda_1, \ldots, \lambda_p)$. Then $X_{\underline{n},A}$, restricted to $\mathrm{int}(\Gamma_{\underline{n}})$, is Hamiltonian with Hamiltonian function*

$$h(x) = \sum_{\beta=1}^{p} \lambda_\beta \sum_{j\in[\beta]} q_j^\beta \log x_j^\beta\ . \tag{4.3}$$

5. Asymptotic dynamics of polymatrix replicators

Given a polymatrix replicator $X_{\underline{n},A}$, the edges and vertices of the polytope $\Gamma_{\underline{n}}$ form a heteroclinic edge network for the associated flow. In this section we recall the technique developed in [3] to analyze the asymptotic dynamics of a flow on a polytope along its heteroclinic edge network. In particular we review the main definitions and results for the polymatrix replicator $X_{\underline{n},A}$ on the polytope $\Gamma_{\underline{n}}$.

The affine support of $\Gamma_{\underline{n}}$ is the smallest affine subspace of $\mathbb{R}^n$ that contains $\Gamma_{\underline{n}}$. It is the subspace $E = E_1 \times \ldots \times E_p$ where for $\alpha = 1, \ldots, p$,

$$E_\alpha := \left\{ x^\alpha \in \mathbb{R}^{[\alpha]} \ :\ \sum_{i\in[\alpha]} x_i^\alpha = 1 \right\}.$$

Following [3, Definition 3.1] we introduce a *defining family* for the polytope $\Gamma_{\underline{n}}$. The affine functions $\{f_i : E \to \mathbb{R}\}_{1\le i\le n}$ where $f_i(x) = x_i$, are a defining family for $\Gamma_{\underline{n}}$ because they satisfy:

(a) $\Gamma_{\underline{n}} = \bigcap_{i\in I} f_i^{-1}([0, +\infty[)$,

(b) $\Gamma_{\underline{n}} \cap f_i^{-1}(0) \neq \emptyset$ for all $i \in \{1, \ldots, n\}$, and

(c) given $J \subseteq \{1, \ldots, n\}$ such that $\Gamma_{\underline{n}} \cap \left(\bigcap_{j\in J} f_j^{-1}(0) \right) \neq \emptyset$, the linear 1-forms $(\mathrm{d}f_j)_p$ are linearly independent at every point $p \in \cap_{j\in J} f_j^{-1}(0)$.

Next we introduce convenient labellings for vertexes, facets and edges of $\Gamma_{\underline{n}}$. Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^n$ and denote by $\mathcal{V}_{\underline{n}}$ the Cartesian product $\mathcal{V}_{\underline{n}} := \prod_{\alpha=1}^p [\alpha]$ which contains $\prod_{\alpha=1}^p n_\alpha$ elements. Each label $\mathrm{j} = (j_1, \ldots, j_p) \in \mathcal{V}_{\underline{n}}$ determines the vertex $v_{\mathrm{j}} := (e_{j_1}, ..., e_{j_p})$ of $\Gamma_{\underline{n}}$. This labelling is one-to-one. The set $\mathcal{F}_{\underline{n}} := \{1, 2, \ldots, n\}$ can be used to label the n facets of $\Gamma_{\underline{n}}$. Each integer $i \in \mathcal{F}_{\underline{n}}$ labels the facet $\sigma_i := \Gamma_{\underline{n}} \cap \{x_i = 0\}$ of $\Gamma_{\underline{n}}$. Edges can be labelled by the set

$\mathcal{E}_{\underline{n}} := \{J \in \mathscr{I}_{\underline{n}} :\ \#J = p+1\ \}$. Given $J \in \mathcal{E}_{\underline{n}}$ there exists a unique (unordered) pair of labels $\mathrm{j}_1, \mathrm{j}_2 \in \mathcal{V}_{\underline{n}}$ such that J is the union of the strategies in j_1 and j_2. The label J determines the edge $\gamma_J := \{t v_{\mathrm{j}_1} + (1-t) v_{\mathrm{j}_2} \colon 0 \leq t \leq 1\}$. Again the correspondence $J \mapsto \gamma_J$ between labels $J \in \mathcal{E}_{\underline{n}}$ and edges of $\Gamma_{\underline{n}}$ is one-to-one.

Given a vertex v of $\Gamma_{\underline{n}}$, we denote by F_v and E_v respectively the sets of facets and edges of $\Gamma_{\underline{n}}$ that contain v. Given $\mathrm{j} = (j_1, \ldots, j_p) \in \mathcal{V}_{\underline{n}}$

$$F_{v_{\mathrm{j}}} = \{\sigma_i :\ i \in \mathcal{F}_{\underline{n}} \setminus \{j_1, \ldots, j_p\}\}$$

and this set of facets contains exactly $n - p = \dim(\Gamma_{\underline{n}})$ elements.

Triples in

$$C := \{\, (v, \gamma, \sigma) \in V \times E \times F :\ \gamma \cap \sigma = \{v\}\,\},$$

are called *corners*. Any pair of elements in a corner uniquely determines the third one. Therefore, sometimes we will shortly refer to a corner (v, γ, σ) as (v, γ) or (v, σ). An edge γ with end-points v, v' determines two corners (v, γ, σ) and (v', γ, σ'), called the *end corners of* γ. The facets σ, σ' are referred to as the *opposite facets of* γ.

Remark 5.1. In a small neighbourhood of a given vertex $v = v_{\mathrm{j}}$, where $\mathrm{j} = (j_1, \ldots, j_p) \in \mathcal{V}_{\underline{n}}$, the affine functions $f_k : \Gamma_{\underline{n}} \to \mathbb{R}$, $f_k(x) := x_k$, with $k \in \mathcal{F}_{\underline{n}} \setminus \{j_1, \ldots, j_p\}$, can be used as a coordinate system for $\Gamma_{\underline{n}}$.

Given a polymatrix replicator $X_{\underline{n},A}$ and a facet σ_i with $i \in [\alpha]$, $\alpha \in \{1, \ldots, p\}$, the i^{th} component of $X_{\underline{n},A}$ is given by

$$\mathrm{d}f_i(X_{\underline{n},A}) = x_i \left((A\,x)_i - \sum_{\beta=1}^{p} (x^\alpha)^T A^{\alpha,\beta} x^\beta \right).$$

Definition 5.2. A polymatrix replicator $X_{\underline{n},A}$ is called regular if for any $i \in \mathcal{F}_{\underline{n}}$, the function $H_i : \Gamma_{\underline{n}} \to \mathbb{R}$,

$$H_i(x) := f_i(x)^{-1}\, \mathrm{d}f_i(X_{\underline{n},A}(x)) = (A\,x)_i - \sum_{\beta=1}^{p} (x^\alpha)^T A^{\alpha,\beta} x^\beta$$

is not identically zero along σ_i.

Clearly generic polymatrix replicators are regular. Using the concept of order of a vector field along a facet [3, Definition 4.2], $X_{\underline{n},A}$ is regular if and only if all facets of $\Gamma_{\underline{n}}$ have order 1. For the sake of simplicity we will assume from now on that all polymatrix replicator vector fields are regular.

Definition 5.3. The *skeleton character* of polymatrix replicator $X_{\underline{n},A}$ is defined to be the matrix $\chi := (\chi^v_\sigma)_{(v,\sigma_{k_\alpha}) \in V \times F}$ where

$$\chi^v_\sigma := \begin{cases} -H_\sigma(v), & v \in \sigma \\ 0 & \text{otherwise} \end{cases}$$

where H_σ stands for H_i when $\sigma = \sigma_i$ with $i \in \mathcal{F}_{\underline{n}}$. For a fixed vertex v, the vector $\chi^v := (\chi^v_\sigma)_{\sigma \in F}$ is referred to as the *skeleton character* at v.

Remark 5.4. Given a corner (v,γ,σ) of $\Gamma_{\underline{n}}$, $H_\sigma(v)$ is the eigenvalue of the tangent map $(\mathrm{d}X_{\underline{n},A})_v$ along the eigen-direction parallel to γ.

Proposition 5.5. *If $X_{\underline{n},A}$ is a regular polymatrix replicator for every vertex $v = v_{\mathrm{j}}$ with label $\mathrm{j} = (j_1,\ldots,j_p) \in \mathcal{V}_{\underline{n}}$, and every facet $\sigma = \sigma_i$ with $i \in \mathcal{F}_{\underline{n}}$ and $i \in [\alpha]$ the skeleton character of $X_{\underline{n},A}$ is given by*

$$\chi_\sigma^v = \begin{cases} \sum_{\beta=1}^{p} (a_{j_\alpha j_\beta} - a_{i j_\beta}) & \text{if } v \in \sigma \\ 0 & \text{otherwise}\,. \end{cases}$$

Proof. Straightforward calculation. □

Remark 5.6. For a given corner (v,γ,σ) of $\Gamma_{\underline{n}}$,

- if $\chi_\sigma^v < 0$ then v is the α-limit of an orbit in γ, and
- if $\chi_\sigma^v > 0$ then v is the ω-limit of an orbit in γ.

Let γ be an edge with end-points v and v' and opposite facets σ and σ', respectively. This means that (v,γ,σ) and (v',γ,σ') are corners of $\Gamma_{\underline{n}}$. If $X_{\underline{n},A}$ does not have singularities in $\mathrm{int}(\gamma)$, then $\mathrm{int}(\gamma)$ consists of a single heteroclinic orbit with α-limit v and ω-limit v' if and only if $\chi_\sigma^v < 0$ and $\chi_{\sigma'}^{v'} > 0$. This type of edges will be referred to as *flowing edges*. The vertices $v = s(\gamma)$ and $v' = t(\gamma)$ are respectively called the *source* and *target* of the flowing edge γ and we will write $v \xrightarrow{\gamma} v'$ to express it.

Given $v = v_{\mathrm{j}}$ with $\mathrm{j} = (j_1,\ldots,j_p) \in \mathcal{V}_{\underline{n}}$ consider the vertex neighbourhood

$$N_v := \{q \in \Gamma_{\underline{n}} :\ 0 \le f_k(q) \le 1,\ \forall k \in \mathcal{F}_{\underline{n}} \backslash \{j_1,\ldots,j_p\}\ \}.$$

Rescaling the defining functions f_k we may assume these neighbourhoods are pairwise disjoint. See Remark 5.1.

For any edge γ with end-points v and v' we define a tubular neighbourhood connecting N_v to $N_{v'}$ by

$$N_\gamma := \{q \in \Gamma_{\underline{n}} \backslash (N_v \cup N_{v'}) :\ 0 \le f_k(q) \le 1,\ \forall\, k \in \mathcal{F}_{\underline{n}} \text{ with } \gamma \subset \sigma_k\}.$$

Again we may assume that these neighbourhoods are pairwise disjoint between themselves. Finally we define the *edge skeleton's tubular neighbourhood* of $\Gamma_{\underline{n}}$ to be

$$N_{\Gamma_{\underline{n}}} := (\cup_{v\in V} N_v) \cup (\cup_{\gamma \in E} N_\gamma). \tag{5.1}$$

The next step is to define the rescaling map $\Psi_\epsilon^{\underline{n},A}$ on $N_{\Gamma_{\underline{n}}} \backslash \partial\Gamma_{\underline{n}}$. See [3, Definition 5.2]. We will write f_σ to denote the affine function f_k associated with the facet $\sigma = \sigma_k$ with $k \in \mathcal{F}_{\underline{n}}$.

Definition 5.7. Let $\epsilon > 0$ be a small parameter. The ϵ-rescaling coordinate system

$$\Psi_\epsilon^{\underline{n},A} : N_{\Gamma_{\underline{n}}} \backslash \partial\Gamma_{\underline{n}} \to \mathbb{R}^F$$

maps $q \in N_{\Gamma_{\underline{n}}}$ to $y := (y_\sigma)_{\sigma\in F}$ where

- if $q \in N_v$ for some vertex v:

$$y_\sigma = \begin{cases} -\epsilon^2 \log f_\sigma(q) & \text{if} \quad v \in \sigma \\ 0 & \text{if} \quad v \notin \sigma \end{cases}$$

- if $q \in N_\gamma$ for some edge γ:

$$y_\sigma = \begin{cases} -\epsilon^2 \log f_\sigma(q) & \text{if} \quad \gamma \subset \sigma \\ 0 & \text{if} \quad \gamma \not\subset \sigma \end{cases}$$

We now turn to the space where these rescaling coordinates take values. For a given vertex $v \in V$ we define

$$\Pi_v := \{ (y_\sigma)_{\sigma \in F} \in \mathbb{R}_+^F : \; y_\sigma = 0, \quad \forall \sigma \notin F_v \} \tag{5.2}$$

where $\mathbb{R}_+ = [0, +\infty)$. Since $\{f_\sigma\}_{\sigma \in F_v}$ is a coordinate system over N_v and the function $h : (0, 1] \to [0, +\infty)$, $h(x) := -\log x$, is a diffeomorphism, the restriction $\Psi_\epsilon^{\underline{n},A} : N_v \backslash \partial\Gamma_{\underline{n}} \to \Pi_v$ is also a diffeomorphism denoted by $\Psi_{\epsilon,v}^{\underline{n},A}$.

If γ is an edge connecting two corners (v, σ) and (v', σ'), $F_v \cap F_{v'} = \{\sigma \in F : \gamma \subset \sigma\}$ and we define

$$\Pi_\gamma := \{ (y_\sigma)_{\sigma \in F} \in \mathbb{R}_+^F : \; y_\sigma = 0 \quad \text{when}\, \gamma \not\subset \sigma \} \,. \tag{5.3}$$

Then $\Psi_\epsilon^{\underline{n},A}(N_\gamma \backslash \partial\Gamma_{\underline{n}}) = \Pi_\gamma = \Pi_v \cap \Pi_{v'}$ has dimension $d-1$ while $\Pi_v = \Psi_{\epsilon.v}^{\underline{n},A}(N_v \backslash \partial\Gamma_{\underline{n}})$ has dimension d. In particular the map $\Psi_{\epsilon,v}^{\underline{n},A}$ is not injective over N_γ. See Figure 4.

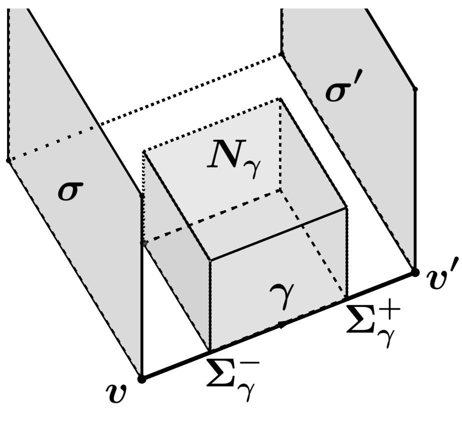

FIGURE 4. An edge connecting two corners.

Definition 5.8. The *dual cone* of $\Gamma_{\underline{n}}$ is defined to be

$$\mathcal{C}^*(\Gamma_{\underline{n}}) := \bigcup_{v \in V} \Pi_v \,,$$

where Π_v is the sector in (5.2).

Hence $\Psi_\epsilon^{\underline{n},A} : N_{\Gamma_{\underline{n}}} \backslash \partial\Gamma_{\underline{n}} \to \mathcal{C}^*(\Gamma_{\underline{n}})$.

Denote by $\{\varphi^t_{\underline{n},A} : \Gamma_{\underline{n}} \to \Gamma_{\underline{n}}\}_{t\in\mathbb{R}}$ the flow of the vector field $X_{\underline{n},A}$. Given a flowing edge γ with source $v = s(\gamma)$ and target $v' = t(\gamma)$ we introduce the cross-sections

$$\Sigma_\gamma^- := (\Psi_{v,\epsilon}^{\underline{n},A})^{-1}(\mathrm{int}(\Pi_\gamma)) \quad \text{and} \quad \Sigma_\gamma^+ := (\Psi_{v',\epsilon}^{\underline{n},A})^{-1}(\mathrm{int}(\Pi_\gamma))$$

transversal to the flow $\varphi^t_{\underline{n},A}$. The sets Σ_γ^- and Σ_γ^+ are inner facets of the tubular neighbourhoods N_v and $N_{v'}$ respectively. Let $\mathscr{D}_\gamma$ be the set of points $x \in \Sigma_\gamma^-$ such that the forward orbit $\{\varphi^t_{\underline{n},A}(x) \colon t > 0\}$ has a first transversal intersection with Σ_γ^+. The global Poincaré map

$$P_\gamma : \mathscr{D}_\gamma \subset \Sigma_\gamma^- \to \Sigma_\gamma^+$$

is defined by $P_\gamma(x) := \varphi^{\tau(x)}_{\underline{n},A}(x)$, where

$$\tau(x) = \min\{\, t > 0 : \; \varphi^t_{\underline{n},A}(x) \in \Sigma_\gamma^+ \,\} \;.$$

To simplify some of the following convergence statements we use the terminology in [3, Definition 5.5].

Definition 5.9. Suppose we are given a family of functions F_ϵ with varying domains $\mathcal{U}_\epsilon$. Let F be another function with domain $\mathcal{U}$. Assume that all these functions have the same target and source spaces, which are assumed to be linear spaces. We will say that $\lim_{\epsilon\to 0^+} F_\epsilon = F$ *in the C^k topology*, to mean that:

(1) *domain convergence*: for every compact subset $K \subseteq \mathcal{U}$, we have $K \subseteq \mathcal{U}_\epsilon$ for every small enough $\epsilon > 0$, and
(2) *uniform convergence on compact sets*:

$$\lim_{\epsilon\to 0^+} \max_{0\le i\le k} \sup_{u\in K} \left| \, D^i \left[F_\epsilon(u) - F(u)\right] \right| \;=\; 0 \;.$$

Convergence in the C^∞ topology means convergence in the C^k topology for all $k \geq 1$. If F_ϵ is a composition of two or more mappings then its domain should be understood as the composition domain.

Let now

$$\Pi_\gamma(\epsilon) := \{\, y \in \Pi_\gamma : \; y_\sigma \geq \epsilon \quad \text{whenever} \quad \gamma \subset \sigma\}, \tag{5.4}$$

and define

$$F_\gamma^\epsilon := \Psi_{v',\epsilon}^{\underline{n},A} \circ P_\gamma \circ (\Psi_{v,\epsilon}^{\underline{n},A})^{-1}.$$

Notice that $\lim_{\epsilon\to 0} \Pi_\gamma(\epsilon) = \mathrm{int}(\Pi_\gamma)$.

Lemma 5.10. *For a given $k \geq 1$, there exists a number r such that the following limit holds in the C^k topology,*

$$\lim_{\epsilon\to 0^+} F^\epsilon_{\gamma|_{\mathcal{U}_\gamma^\epsilon}} = \mathrm{id}_{\Pi_\gamma},$$

where $\mathcal{U}_\gamma^\epsilon \subset \Pi_\gamma(\epsilon^r)$ is the domain of F_γ^ϵ.

Proof. See [3, Lemma 7.2]. □

Hence, since the global Poincaré maps converge towards the identity map as we approach the heteroclinic orbit, the asymptotic behaviour of the flow is solely determined by local Poincaré maps.

From Definition 5.3, for any vertex v, the vector χ^v is tangent to Π_v, in the sense that χ^v belongs to the linear span of the sector Π_v. Let

$$\Pi_v(\epsilon) := \{\, y \in \Pi_v : \; y_\sigma \geq \epsilon \quad \text{for all} \quad \sigma \in F_v \,\} \tag{5.5}$$

Using the notation of Definition 5.3 we have

Lemma 5.11. *We have*

$$(\Psi^{\underline{n},A}_{v,\epsilon})_* X_{\underline{n},A} = \epsilon^2 \left(\tilde{X}^\epsilon_{v,\sigma} \right)_{\sigma \in F} ,$$

where

$$\tilde{X}^\epsilon_{v,\sigma}(y) := \begin{cases} -H_\sigma \left((\Psi^{\underline{n},A}_{v,\epsilon})^{-1}(y) \right) & \text{if} \;\; \sigma \in F_v \\ 0 & \text{if} \;\; \sigma \notin F_v \end{cases} ,$$

Moreover, given $k \geq 1$ there exists $r > 0$ such that the following limit holds in the C^k topology

$$\lim_{\epsilon \to 0} (\tilde{X}^\epsilon_v)_{|\Pi_v(\epsilon^r)} = \chi^v \,.$$

Proof. See [3, Lemma 5.6]. □

Consider a vertex v with an incoming flowing-edge $v_* \xrightarrow{\gamma} v$ and an outgoing flowing-edge $v \xrightarrow{\gamma'} v'$. Denote by σ_* the facet opposed to γ' at v. We define the sector

$$\Pi_{\gamma,\gamma'} := \left\{ y \in \operatorname{int}(\Pi_\gamma) : \; y_\sigma - \frac{\chi^v_\sigma}{\chi^v_{\sigma_*}} \, y_{\sigma_*} > 0, \, \forall \sigma \in F_v, \; \sigma \neq \sigma_* \right\} \tag{5.6}$$

and the linear map $L_{\gamma,\gamma'} : \Pi_{\gamma,\gamma'} \to \Pi_{\gamma'}$ by

$$L_{\gamma,\gamma'}(y) := \left(y_\sigma - \frac{\chi^v_\sigma}{\chi^v_{\sigma_*}} \, y_{\sigma_*} \right)_{\sigma \in F} . \tag{5.7}$$

Notice that $\Pi_{\gamma'} = \{y \in \Pi_v : y_{\sigma_*} = 0\}$ as well as Π_γ are facets to Π_v.

Proposition 5.12. *The sector $\Pi_{\gamma,\gamma'}$ consists of all points $y \in \operatorname{int}(\Pi_\gamma)$ which can be connected to some point $y' \in \operatorname{int}(\Pi_{\gamma'})$ by a line segment inside the ray $\{\, y + t\chi^v : t \geq 0 \,\}$. Moreover, if $y \in \Pi_{\gamma,\gamma'}$ then the other endpoint is $y' = L_{\gamma,\gamma'}(y)$.*

Proof. See [3, Proposition 6.4]. □

Given flowing-edges γ and γ' such that $t(\gamma) = s(\gamma') = v$ we denote by $\mathscr{D}_{\gamma,\gamma'}$ the set of points $x \in \Sigma_{v,\gamma}$ such that the forward orbit $\{\varphi^t_{\underline{n},A}(x) \colon t \geq 0\}$ has a first transversal intersection with $\Sigma_{v,\gamma'}$. The local Poincaré map

$$P_{\gamma,\gamma'} : \mathscr{D}_{\gamma,\gamma'} \subset \Sigma^+_\gamma \to \Sigma^-_{\gamma'}$$

is defined by $P_{\gamma,\gamma'}(x) := \varphi^{\tau(x)}_{\underline{n},A}(x)$, where

$$\tau(x) := \min\{\, t > 0 : \; \varphi^t_{\underline{n},A}(x) \in \Sigma^-_{\gamma'} \,\} \,.$$

Lemma 5.13. *Let $\mathcal{U}^{\epsilon}_{\gamma,\gamma'} \subset \Pi_\gamma(\epsilon^r)$ be the domain of the map*

$$F^{\epsilon}_{\gamma,\gamma'} := \Psi^{\underline{n},A}_{v,\epsilon} \circ P_{\gamma,\gamma'} \circ (\Psi^{\underline{n},A}_{v,\epsilon})^{-1}.$$

Then for a given $k \geq 1$ there exist $r > 0$ such that

$$\lim_{\epsilon \to 0^+} \left(F^{\epsilon}_{\gamma,\gamma'}\right)_{|\mathcal{U}^{\epsilon}_{\gamma,\gamma'}} = L_{\gamma,\gamma'}$$

in the C^k topology.

Proof. See [3, Lemma 7.5]. □

Given a chain of flowing-edges

$$v_0 \xrightarrow{\gamma_0} v_1 \xrightarrow{\gamma_1} v_2 \longrightarrow \ldots \longrightarrow v_m \xrightarrow{\gamma_m} v_{m+1}$$

the sequence $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$ is called a *heteroclinic path*, or a *heteroclinic cycle* when $\gamma_m = \gamma_0$.

Definition 5.14. Given a heteroclinic path $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$:

1) The *Poincaré map* of a polymatrix replicator $X_{\underline{n},A}$ along ξ is the composition

$$P_\xi := (P_{\gamma_m} \circ P_{\gamma_{m-1},\gamma_m}) \circ \ldots \circ (P_{\gamma_1} \circ P_{\gamma_0,\gamma_1}),$$

whose domain is denoted by $\mathcal{U}_\xi$.

2) The *skeleton flow map (of χ) along ξ* is the composition map $\pi_\xi : \Pi_\xi \to \Pi_{\gamma_m}$ defined by

$$\pi_\xi := L_{\gamma_{m-1},\gamma_m} \circ \ldots \circ L_{\gamma_0,\gamma_1} ,$$

whose domain is

$$\Pi_\xi := \mathrm{int}(\Pi_{\gamma_0}) \cap \bigcap_{j=1}^{m} (L_{\gamma_*,\gamma_j} \circ \ldots \circ L_{\gamma_0,\gamma_1})^{-1} \mathrm{int}(\Pi_{\gamma_j}) .$$

The previous lemmas 5.10 and 5.13 imply that given a heteroclinic path ξ, the asymptotic behaviour of the Poincaré map P_ξ along ξ is given by the Poincaré map π_ξ of χ.

Proposition 5.15. *Let $\mathcal{U}^{\epsilon}_{\xi}$ be the domain of the map*

$$F^{\epsilon}_{\xi} := \Psi^{\underline{n},A}_{v_m,\epsilon} \circ P_\xi \circ \left(\Psi^{\underline{n},A}_{v_0,\epsilon}\right)^{-1}$$

from $\Pi_{\gamma_0}(\epsilon^r)$ into $\Pi_{\gamma_m}(\epsilon^r)$. Then

$$\lim_{\epsilon \to 0^+} \left(F^{\epsilon}_{\xi}\right)_{|\mathcal{U}^{\epsilon}_{\xi}} = \pi_\xi$$

in the C^k topology.

Proof. See [3, Proposition 7.7]. □

To analyse the dynamics of the flow of the skeleton vector field χ we introduce the concept of *structural set* and its associated skeleton flow map. See [3, Definition 6.8].

Definition 5.16. A non-empty set of flowing-edges S is said to be a *structural set* for χ if every heteroclinic cycle contains an edge in S.

Structural sets are in general not unique. We say that a heteroclinic path $\xi = (\gamma_0, \dots, \gamma_m)$ is an S-branch if
(1) $\gamma_0, \gamma_m \in S$,
(2) $\gamma_j \notin S$ for all $j = 1, \dots, m-1$.
Denote by $\mathscr{B}_S(\chi)$ the set of all S-branches.

Definition 5.17. The *skeleton flow map* $\pi_S : D_S \to \Pi_S$ is defined by

$$\pi_S(y) := \pi_\xi(y) \quad \text{for all } \; y \in \Pi_\xi,$$

where

$$D_S := \cup_{\xi \in \mathscr{B}_S(\chi)} \Pi_\xi \quad \text{and} \quad \Pi_S := \cup_{\gamma \in S} \Pi_\gamma.$$

The reader should picture $\pi_S : D_S \to \Pi_S$ as the first return map of the piecewise linear flow of χ on $\mathcal{C}^*(\Gamma_{\underline{n}})$ to the system of cross-sections Π_S. The following, see [3, Proposition 6.10], provides a sufficient condition for the skeleton flow map π_S to be a closed dynamical system.

Proposition 5.18. *Given a skeleton vector field χ on $\mathcal{C}^*(\Gamma_{\underline{n}})$ with a structural set S, assume*
(1) every edge of $\Gamma_{\underline{n}}$ is either neutral or a flowing-edge,
(2) every vertex v is of saddle type, i.e., $\chi^v_{\sigma_1} \chi^v_{\sigma_2} < 0$ for some facets $\sigma_1, \sigma_2 \in F_v$.
Then

$$\hat{D}_S := \bigcap_{n \in \mathbb{Z}} (\pi_S)^{-n}(D_S)$$

is a Baire space with full Lebesgue measure in Π_S and $\pi_S : \hat{D}_S \to \hat{D}_S$ is a homeomorphism.

Given a structural set S any orbit of the flow $\varphi^t_{\underline{n},A}$ that shadows some heteroclinic cycle must intersect the cross-sections $\cup_{\gamma \in S} \Sigma^+_\gamma$ recurrently. The following map encapsulates the semi-global dynamics of these orbits.

Definition 5.19. Given $X_{\underline{n},A}$, let S be a structural set of its skeleton vector field. We define $P_S : \mathcal{U}_S \subset \Sigma_S \to \Sigma_S$ setting $\Sigma_S := \cup_{\gamma \in S} \Sigma^+_\gamma$, $\mathcal{U}_S := \cup_{\xi \in \mathscr{B}_S(\chi)} \mathcal{U}_\xi$ and $P_S(p) := P_\xi(p)$ for all $p \in \mathcal{U}_\xi$. The domain components $\mathcal{U}_\xi$ and $\mathcal{U}_{\xi'}$ are disjoint for branches $\xi \neq \xi'$ in $\mathscr{B}_S(\chi)$.

Up to a time re-parametrization, the map $P_S : D_S \subset \Sigma_S \to \Sigma_S$ embeds in the flow $\varphi^t_{(\underline{n},A)}$. In this sense the dynamics of P_S encapsulates the qualitative behaviour of the flow φ^t_X of X along the edges of $\Gamma_{\underline{n}}$.

Theorem 5.20. *Let $X_{\underline{n},A}$ be a regular polymatrix replicator with skeleton vector field χ. If S is a structural set of χ then*

$$\lim_{\epsilon \to 0^+} \Psi_\epsilon \circ P_S \circ (\Psi_\epsilon)^{-1} = \pi_S$$

in the C^∞ topology, in the sense of Definition 5.9.

Proof. See [3, Theorem 7.9]. $\square$

6. Hamiltonian character of the asymptotic dynamics

In this section we discuss the Poisson geometric properties of the Poincaré maps π_ξ in the case of Hamiltonian polymatrix replicator equations.

Given a generic Hamiltonian polymatrix replicator, $X_{\underline{n},A_0}$, we study its asymptotic Poincaré maps, proving that they are Poisson maps.

Let $X_{\underline{n},A}$ be a conservative polymatrix replicator, q a formal equilibrium, A_0 and D as in Definition 4.4, and

$$h(x) = \sum_{\beta=1}^{p} \sum_{j\in[\beta]} \lambda_\beta q_j^\beta \log x_j^\beta \tag{6.1}$$

its Hamiltonian function as in Theorem 4.5. The Hamiltonian (6.1) belongs to a class of prospective constants of motion for vector fields on polytopes discussed in [3, Section 8]. Since the polymatrix replicator is fixed we drop superscript $(\underline{n}, A)$ and use $\Psi_{v,\epsilon}$ for the rescaling coordinate systems defined in Definition 5.7. The following proposition gives us the asymptotic constant of motion, on the dual cone, associated to h.

Proposition 6.1. *Let $\eta : \mathcal{C}^*(\Gamma_{\underline{n}}) \to \mathbb{R}$ be defined by*

$$\eta(y) := \sum_{\beta=1}^{p} \sum_{j\in[\beta]} \lambda_\beta q_j^\beta y_j^\beta . \tag{6.2}$$

(1) $\eta = \lim_{\epsilon\to 0^+} \epsilon^2 h \circ (\Psi_{v,\epsilon})^{-1}$ over $\mathrm{int}(\Pi_v)$ for any vertex v, with convergence in the C^∞ topology.
(2) $d\eta = \lim_{\epsilon\to 0^+} \epsilon^2 \left[(\Psi_{v,\epsilon})^{-1}\right]^* (dh)$ over $\mathrm{int}(\Pi_v)$ for any vertex v, with convergence in the C^∞ topology.
(3) Since h is invariant under the flow of $X_{\underline{n},A}$, i.e., $dh(X_{\underline{n},A}) \equiv 0$, the function η is invariant under the skeleton flow of χ, i.e., $d\eta(\chi) \equiv 0$.

Proof. See [3, Proposition 8.2]. $\square$

We will use the following family of coordinate charts for the Poisson manifold $(\mathrm{int}(\Gamma_{\underline{n}}), \pi_{A_0})$ where π_{A_0} is defined at (4.2).

Definition 6.2. Given a vertex $v = (e_{j_1}, ..., e_{j_p})$ of $\Gamma_{\underline{n}}$, we set $\hat{x}_\alpha := (x_{\hat{k}}^\alpha)_{\hat{k}\in[\alpha]\setminus\{j_\alpha\}}$ and $\hat{x} := (\hat{x}^\alpha)_\alpha$, and define the projection map

$$P_v : \mathrm{int}(N_v) \to (\mathbb{R}^{n_1-1} \times \ldots \times \mathbb{R}^{n_p-1}), \quad P_v(x) := \hat{x}.$$

Clearly, P_v is a diffeomorphism onto its image $(0,1)^{n-p}$. The inverse map $\psi_v := P_v^{-1}$ can be regarded as a local chart for the manifold $\mathrm{int}(\Gamma_{\underline{n}})$.

Remark 6.3. The projection map P_v extends linearly to $\mathbb{R}^n$ and it is represented by $(n-p)\times n$ block diagonal matrix

$$P_v = \operatorname{diag}(P_v^1, \ldots, P_v^p)$$

where P_v^α, $\alpha = 1,\ldots,p$, is the constant $(n_\alpha - 1)\times n_\alpha$ matrix obtained removing the row j_α from the identity matrix.

Lemma 6.4. *Consider the Poisson manifold* $(\operatorname{int}(\Gamma_{\underline{n}}), \pi_{A_0})$ *where* π_{A_0} *is defined at* (4.2)*. Then for any vertex* v*, the matrix representation of* π_{A_0} *in the local chart* ψ_v *is*

$$\pi_{A_0}^{\sharp_v}(\hat{x}) = (-1)\, P_v\, T_x\, D_x\, A_0\, D_x\, T_x^t\, P_v^t, \tag{6.3}$$

where D_x *and* T_x *are as defined at* (4.2)*.*

Proof. Notice that $\pi_{A_0}^{\sharp_v}(\hat{x}) := [\{x_{\hat{k}}^\alpha, x_{\hat{l}}^\beta\}]$ with $\alpha,\beta = 1,\ldots,p$ and $\hat{k}\in[\alpha], \hat{l}\in[\beta]$. □

We used the notation $\sharp_v$ instead of $\sharp$ to make it clear that the representing matrix is w.r.t. the local chart ψ_v. The following trivial Lemma gives us the differential of the ϵ-rescaling map $\Psi_{v,\epsilon}$, which defined in Definition 5.7, in the coordinate chart ψ_v.

Lemma 6.5. *The differential of the diffeomorphism*

$$\Psi_{v,\epsilon}\circ\psi_v : P_v(\operatorname{int}(N_v)) \to \operatorname{int}(\Pi_v)$$

is

$$D(\Psi_{v,\epsilon}\circ\psi_v)_{\hat{x}} = -\epsilon^2 \operatorname{diag}(D^1(\Psi_{v,\epsilon}\circ\phi_v)_{\hat{x}^1}, \ldots, D^p(\Psi_{v,\epsilon}\circ\phi_v)_{\hat{x}^p}),$$

where $D^\alpha(\Psi_{v,\epsilon}\circ\psi_v)_{\hat{x}^\alpha}$ $\alpha = 1,\ldots,p$ *is given by*

$$\operatorname{diag}((x_1^\alpha)^{-\nu_{1\alpha}}, \ldots, (x_{j_\alpha-1}^\alpha)^{-\nu_{j_\alpha-1}}, (x_{j_\alpha+1}^\alpha)^{-\nu_{j_\alpha+1}}, \ldots, (x_{n_\alpha}^\alpha)^{-\nu_{n_\alpha}}).$$

We push forward, by the diffeomorphism $\Psi_{v,\epsilon}\circ\psi_v$, the Poisson structure $\pi_{A_0}^{\sharp_v}$ defined on $P_v(\operatorname{int}(N_v))$ to $\operatorname{int}(\Pi_v)$. The following lemma provides the matrix representation of the push forwarded Poisson structure. In order to simplify the notation we set

$$\mathbb{J}(\hat{x}) := D(\Psi_{v,\epsilon}\circ\psi_v)_{\hat{x}}\, P_v\, T_x\, D_x \tag{6.4}$$

and for every $\alpha = 1,\ldots,p$

$$\mathbb{J}_\alpha(\hat{x}^\alpha) := D^\alpha(\Psi_{v,\epsilon}\circ\psi_v)_{\hat{x}^\alpha}\, P_v^\alpha\, T_x^\alpha\, D_{x^\alpha}. \tag{6.5}$$

Notice that $\mathbb{J}(\hat{x}) = \operatorname{diag}(\mathbb{J}_1(\hat{x}^1), \ldots, \mathbb{J}_p(\hat{x}^p))$.

Lemma 6.6. *The diffeomorphism* $\Psi_{v,\epsilon}\circ\psi_v$ *pushes forward the Poisson structure* $\pi_{A_0}^{\sharp_v}$ *to the Poisson structure* $\pi_{A_0,\epsilon}^{\sharp_v}$ *on* $\operatorname{int}(\Pi_v)$ *where*

$$\pi_{A_0,\epsilon}^{\sharp_v}(y) = (-1)(\mathbb{J}A_0\mathbb{J}^t)\circ(\Psi_{v,\epsilon}\circ\psi_v)^{-1}(y). \tag{6.6}$$

Proof. See Definition 3.1 and Remark 3.2. □

If all the faces $\sigma \in F_v$ have order one, the Poisson structure $\pi^{\sharp_v}_{A_0,\epsilon}$ is asymptotically equivalent to a linear Poisson structure. Let

$$E_v = \mathrm{diag}(E_v^1, \ldots, E_v^p), \tag{6.7}$$

be $((n-p)\times n)$-matrix defined by diagonal blocks E_v^α, for $\alpha = 1,..,p$, where α^{th} block is the $((n_\alpha - 1)\times n_\alpha)$-matrices in which the column j_α is equal to $\mathbb{1}_{n_\alpha - 1}$ and every other columns $k_\alpha \neq j_\alpha$ is equal to $-e_{k_\alpha} \in \mathbb{R}^{n_\alpha - 1}$.

Lemma 6.7. *For a given vertex $v = (e_{j_1}, ..., e_{j_p})$, let E_v be the matrix defined at* (6.7) *and $B_v := E_v A_0 E_v^t$. If $\nu_\sigma = 1$ for every $\sigma \in F_v$, then*

$$\lim_{\epsilon\to 0^+} \frac{-1}{\epsilon^2}\mathbb{J}\circ(\Psi_{v,\epsilon}\circ\psi_v)^{-1}(y) = E_v,$$

over $\mathrm{int}(\Pi_v)$ *with convergence in C^∞ topology. Consequently,*

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon^4}\pi^{\sharp_v}_{A_0,\epsilon}(y) = B_v,$$

over $\mathrm{int}(\Pi_v)$ *with convergence in C^∞ topology.*

Proof. A simple calculation shows that for every $\alpha = 1,..,p$

$$\frac{-1}{\epsilon^2}\mathbb{J}_\alpha = D^\alpha_{x,\nu}\begin{pmatrix} (x_1^\alpha - 1) & \ldots & x^\alpha_{j_\alpha - 1} & x^\alpha_{j_\alpha} & x^\alpha_{j_\alpha - 1} & \ldots & x^\alpha_{n_\alpha} \\ & & & \vdots & & & \\ x_1^\alpha & \ldots & (x^\alpha_{j_\alpha - 1} - 1) & x^\alpha_{j_\alpha} & x^\alpha_{j_\alpha + 1} & \ldots & x^\alpha_{n_\alpha} \\ x_1^\alpha, & \ldots & x^\alpha_{j_\alpha - 1} & x^\alpha_{j_\alpha} & (x^\alpha_{j_\alpha + 1} - 1) & \ldots & x^\alpha_{n_\alpha} \\ & & & \vdots & & & \\ x_1^\alpha & \ldots, & x^\alpha_{j_\alpha - 1} & x^\alpha_{j_\alpha} & x^\alpha_{j_\alpha + 1} & \ldots & (x^\alpha_{n_\alpha} - 1) \end{pmatrix},$$

where

$$D^\alpha_{x,\nu} := \mathrm{diag}((x_1^\alpha)^{1-\nu_{1_\alpha}}, \ldots, (x^\alpha_{j_\alpha - 1})^{1-\nu_{j_\alpha - 1}}, (x^\alpha_{j_\alpha + 1})^{1-\nu_{j_\alpha + 1}}, \ldots, (x^\alpha_{n_\alpha})^{1-\nu_{n_\alpha}})$$

Since $\nu_\sigma = 1$ for every $\sigma \in F_v$, for any $\hat{k} \in [\alpha]$ we have

$$\lim_{\epsilon\to 0^+} x^\alpha_{\hat{k}} \circ (\Psi_{v,\epsilon}\circ\psi_v)^{-1}(y) = \lim_{\epsilon\to 0^+} e^{-\frac{y^\alpha_{\hat{k}}}{\epsilon^2}} = 0.$$

Considering that $x^\alpha_{j_\alpha} = 1 - \sum_{\hat{k}\in[\alpha]} x^\alpha_{\hat{k}}$, we get the first claim of the lemma and the second claim is an immediate consequence. □

Figure 5 explains the situation for $\Gamma = \Delta^2$.

Remark 6.8. The same linear Poisson structure $B_v := E_v A_0 E_v^t$ appears in [2, Theorem 3.5].

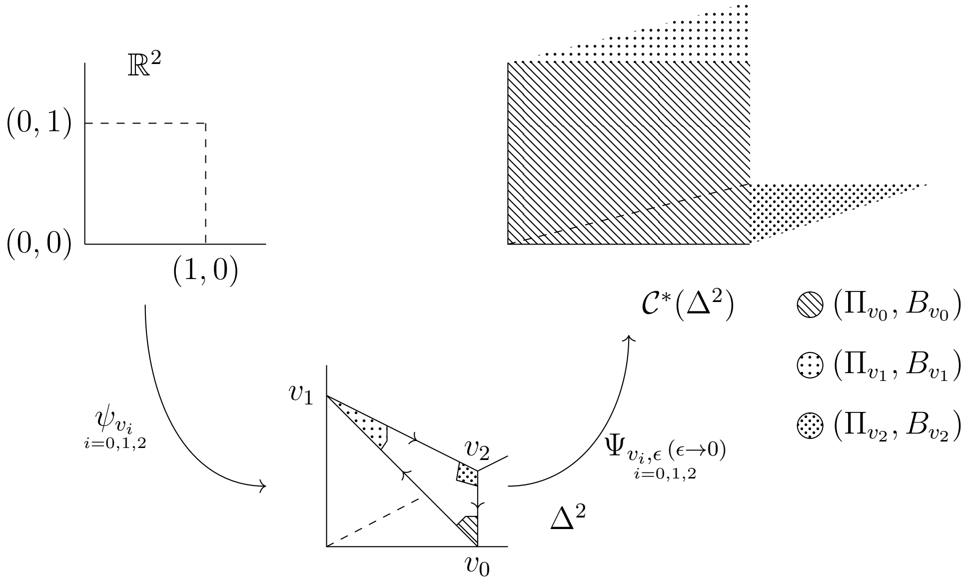

FIGURE 5. Poisson structures on the dual cone.

Lemma 6.9. *For a given vertex $v = (e_{j_1}, ..., e_{j_p})$, let χ^v be the skeleton character of $X_{\underline{n},A}$, as in Definition 5.3. Then*

$$\chi^v = B_v d\eta_v,$$

where η_v is the restriction of function η (defined in (6.2)) to $\mathrm{int}(\Pi_v)$. In other words, χ^v restricted to $\mathrm{int}(\Pi_v)$ is Hamiltonian w.r.t. the constant Poisson structure B_v having η_v as a Hamiltonian function.

Proof. We use the notation $X^v_{(\underline{n},A_0)}(\hat{x}) := (D_x P_v) X_{\underline{n},A}(x)$ for the local expression of the replicator vector field $X_{\underline{n},A}$ in the local chart ψ_v. If we write the function $h(x)$, defined in (6.1), as $h(x) = h(\psi_v \circ P_v(x))$ then

$$d_x h = (P_v)^t d_{\hat{x}}(h \circ \psi_v)(\hat{x}).$$

Notice that $DP_v = P_v$. By Theorem 4.5, $X_{\underline{n},A_0} = \pi_{A_0} dh$. Locally,

$$X^v_{(\underline{n},A_0)}(\hat{x}) = P_v X_{\underline{n},A}(x) = P_v \pi_{A_0} P_v^t d_{\hat{x}}(h \circ \psi_v).$$

Similarly, writing $h \circ \psi_v(\hat{x}) = h \circ \psi_v \circ (\Psi_{v,\epsilon} \circ \psi_v)^{-1} \circ (\Psi_{v,\epsilon} \circ \psi_v)(\hat{x})$ we have

$$d_{\hat{x}}(h \circ \psi_v) = (D_{\hat{x}}(\Psi_{v,\epsilon} \circ \psi_n))^t d_y(h \circ (\Psi_{v,\epsilon})^{-1}).$$

The vector field $\tilde{X}^\epsilon_v$ defined in Lemma 5.11 is

$$\begin{aligned}
\tilde{X}^\epsilon_v &= \frac{1}{\epsilon^2}(D_x(\Psi_{v,\epsilon}) X_{\underline{n},A}) = \frac{1}{\epsilon^2}(D_{\hat{x}}(\Psi_{v,\epsilon} \circ \psi_v) X^v_{(\underline{n},A_0)}) \\
&= \frac{1}{\epsilon^2}(D_{\hat{x}}(\Psi_{v,\epsilon} \circ \psi_v) P_v \pi_{A_0} P_v^t D_{\hat{x}}(\Psi_{v,\epsilon} \circ \psi_n))^t d_y(h \circ (\Psi_{v,\epsilon})^{-1}) \\
&= \frac{1}{\epsilon^4} \pi^{\sharp_v}_{A_0,\epsilon}\Big(\epsilon^2((\Psi_{v,\epsilon})^{-1})^* d_x h\Big)
\end{aligned}$$

In the second equality we used $\psi_v \circ P_v = Id$. Applying Lemma 5.11, Lemma 6.7 and Proposition 6.1 yields the result. Notice that $\Pi_v(\epsilon^r) \subset \mathrm{int}(\Pi_v)$. □

Our aim is to show that for a given heteroclinic path $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$, the skeleton flow map of χ along ξ (see Definition 5.14),

$$\pi_\xi := L_{\gamma_{m-1},\gamma_m} \circ \ldots \circ L_{\gamma_0,\gamma_1} ,$$

restricted to the level set of η, is a Poisson map. Notice that the Poisson structure B_v is only defined in $\mathrm{int}(\Pi_v)$ and neither Π_γ nor $\Pi_{\gamma'}$ are submanifolds of $\mathrm{int}(\Pi_v)$. So we need to define Poisson structures on the sections $\Pi_{\gamma_i,\gamma_{i+1}}$ for all $i = 0, ..., m$.

We start with a single flowing vertex $v_1 = (e_{j_1}, ..., e_{j_p})$ with an incoming flowing-edge $v_0 \xrightarrow{\gamma} v_1$ and an outgoing flowing-edge $v_1 \xrightarrow{\gamma'} v_2$. By construction of $\Gamma_{\underline{n}}$, there exist $1 \le \alpha_0^{v_1}, \alpha_2^{v_1} \le p$, $\hat{k}_0 \in [\alpha]_0^{v_1}$ and $\hat{k}_2 \in [\alpha]_2^{v_1}$ such that

$$v_0 = v_1 + (0, \ldots, e_{\hat{k}_0} - e_{j_{\alpha_0^{v_1}}}, \ldots, 0) \quad \text{and} \quad v_2 = v_1 + (0, \ldots, e_{\hat{k}_2} - e_{j_{\alpha_2^{v_1}}}, \ldots, 0).$$

Remark 6.10. To simply notations, we omit the superscript v_1 from $\alpha_0^{v_1}, \alpha_2^{v_1}$ whenever there is no confusion. Also, denoting $v_0 = (e_{j_1^0}, ..., e_{j_p^0})$, clearly, $\hat{k}_0 = j^0_{\alpha_0^{v_1}}$ and similarly $\hat{k}_2 = j^2_{\alpha_2^{v_1}}$. Again, in order to keep notations as simple as possible we use notations $\hat{k}_0$ and $\hat{k}_1$.

Clearly, $\gamma' = v_1 + s(0, \ldots, e_{\hat{k}_2} - e_{j_{\alpha_0}}, \ldots, 0)$. The opposite facet to γ' at v_1 is $\sigma_* := \{y_{\hat{k}_2}^{\alpha_2} = 0\}$. We also have

$$\begin{aligned}
&\Pi_{v_1} = \{y \in \mathbb{R}^n_+ \,|\, y^1_{j_1} = \ldots = y^p_{j_p} = 0\},\\
&\Pi_{v_r} = \{y \in \mathbb{R}^n_+ \,|\, y^1_{j_1} = \ldots = y^{\alpha_r-1}_{j_{\alpha_r-1}} = y^{\alpha_r}_{\hat{k}_r} = y^{\alpha_r+1}_{j_{\alpha_r+1}} = \ldots = y^p_{j_p} = 0\},\ r = 0, 2\\
&\Pi_\gamma = \{y \in \mathbb{R}^n_+ \,|\, y^1_{j_1} = \ldots = y^{\alpha_0-1}_{j_{\alpha_0-1}} = y^{\alpha_0}_{j_{\alpha_0}} = y^{\alpha_0}_{\hat{k}_0} = y^{\alpha_0+1}_{j_{\alpha_0+1}} = \ldots = y^p_{j_p} = 0\},\\
&\Pi_{\gamma'} = \{y \in \mathbb{R}^n_+ \,|\, y^1_{j_1} = \ldots = y^{\alpha_2-1}_{j_{\alpha_2-1}} = y^{\alpha_2}_{j_{\alpha_2}} = y^{\alpha_2}_{\hat{k}_2} = y^{\alpha_2+1}_{j_{\alpha_2+1}} = \ldots = y^p_{j_p} = 0\}.
\end{aligned} \tag{6.8}$$

The skeleton flow map of χ at vertex v_1 is the linear map $L_{\gamma,\gamma'} : \Pi_{\gamma,\gamma'} \to \Pi_{\gamma'}$ defined by

$$L_{\gamma,\gamma'}(y) := \left(y^\alpha_{\hat{i}_\alpha} - \frac{(\chi^{v_1})^\alpha_{\hat{i}_\alpha}}{(\chi^{v_1})^{\alpha_2}_{\hat{k}_2}}\, y^{\alpha_2}_{\hat{k}_2} \right)_{\alpha,\hat{i}_\alpha} , \tag{6.9}$$

where

$$\Pi_{\gamma,\gamma'} := \left\{ y \in \mathrm{int}(\Pi_\gamma) :\ y^\alpha_{\hat{i}_\alpha} - \frac{(\chi^{v_1})^\alpha_{\hat{i}_\alpha}}{(\chi^{v_1})^{\alpha_2}_{\hat{k}_2}}\, y^{\alpha_2}_{\hat{k}_2} > 0,\ \forall \alpha, \hat{i}_\alpha \ne \alpha_2, \hat{k}_2 \right\}. \tag{6.10}$$

Remark 6.11. In [3], this map is referred to as *Asymptotic Poincaré map*. However, it is not a Poincaré map in the conventional way, since the skeleton vector filed χ is only defined in $\mathrm{int}(\Pi_v)$ for all $v \in F$ and its value on the sections Π_γ is zero. For this reason, we cannot use, directly, the techniques introduced in Section 3. To solve this issue we will use two cosymplectic foliations of $\mathrm{int}(\Pi_v)$.

In fact, we need to restrict ourself to a tubular neighborhood, see Definition 6.12, rather than the entire $\mathrm{int}(\Pi_{v_1})$. For the flowing vertex $v_0 \xrightarrow{\gamma} v_1 \xrightarrow{\gamma'} v_2$, we define the following family of maps

$$L^{\delta}_{\gamma,\gamma'}(y) := \left(y^{\alpha}_{\hat{i}_\alpha} - \frac{\delta (\chi^{v_1})^{\alpha}_{\hat{i}_\alpha}}{(\chi^{v_1})^{\alpha_2}_{\hat{k}_2}} \, y^{\alpha_2}_{\hat{k}_2} \right)_{\alpha, \hat{i}_\alpha}, \quad \delta \in (0,1).$$

Notice that $L_{\gamma,\gamma'}(y) = \phi_{\chi^{v_1}}(t(y), y)$ where $\phi_{\chi^{v_1}}(t,y) = y + t\chi^{v_1}$, is the flow of the skeleton vector field χ^{v_1} and $t(y) := -\frac{y^{\alpha_2}_{\hat{k}_2}}{(\chi^{v_1})^{\alpha_2}_{\hat{k}_2}}$.

Definition 6.12. We denote by

$$T_{\gamma,\gamma'} := \bigcup_{0<\delta<1} L^{\delta}_{\gamma,\gamma'}(\Pi_{\gamma,\gamma'}), \tag{6.11}$$

the tubular neighborhood containing the segments of the flow of χ^{v_1} connecting points in the domain of the asymptotic Poincaré map, $L_{\gamma,\gamma'}$, to their images.

We, now, describe the Poisson structures on $\Pi_{\gamma,\gamma'}$ and $L_{\gamma,\gamma'}(\Pi_{\gamma,\gamma'})$.

Lemma 6.13. *As we adapted in Lemma 6.9, let η_{v_1} be the restriction of function η, defined in (6.2), to $\mathrm{int}(\Pi_{v_1})$. Consider two functions $G^{v_1}_r : T_{\gamma,\gamma'} \to \mathbb{R}$, $r = 0,2$ defined by $G^{v_1}_r(y) = y^{\alpha_r}_{\hat{k}_r}$, then*

1) *Level sets of $(\eta_{v_1}, G^{v_1}_r) : T_{\gamma,\gamma'} \to \mathbb{R}^2$ partition $T_{\gamma,\gamma'}$ into a cosymplectic foliation $\mathcal{F}^{v_1}_r$ i.e. every leaf of $\mathcal{F}^{v_1}_r$ is a cosymplectic submanifold of $(T_{\gamma,\gamma'}, B_{v_1})$. Furthermore, every leaf Σ of this foliation is a level transversal section to χ^{v_1} at every point $x \in \Sigma$.*
2) *For a fixed $r \in \{0,2\}$ the Poincaré map between two given leafs of $\mathcal{F}^{v_1}_r$ is a Poisson map.*
3) *The Poincaré map from one given leaf of $\mathcal{F}^{v_1}_0$ to a leaf of $\mathcal{F}^{v_1}_1$ is a Poisson map.*

Proof. For $r = 0,2$, we have that

$$\{\eta_{v_1}, G^{v_1}_r\} = X_{\eta_{v_1}}(dG^{v_1}_r) = \chi^{v_1}(dG^{v_1}_r) = (\chi^{v_1})^{\alpha_r}_{\hat{k}_r}.$$

Since v_1 is a flowing vertex both $(\chi^{v_1})^{\alpha_r}_{\hat{k}_r}$, $r = 0,2$ are nonzero. This means that η_{v_1} and $G^{v_1}_r$ are second class constraints, s their level sets are cosymplectic submanifolds, see Definition 3.4. The fact that Σ is a level transversal section is clear. Proposition 3.11 yields the second and third claims of the lemma. ☐

The Poincaré map mentioned in Item (2) of Lemma 6.13 is a translation. Fixing $r \in \{0,2\}$, consider two level set $(\eta_{v_1}, G^{v_1}_r)^{-1}(c, d_j)$, $j = 0,2$, then the Poincaré map between them is the translation

$$P(y) = \phi_{\chi^{v_1}}\left(\frac{d_2 - d_0}{(\chi^{v_1})^{\alpha_r}_{\hat{k}_r}}, y\right) = \left(\frac{d_2 - d_0}{(\chi^{v_1})^{\alpha_r}_{\hat{k}_r}}\right) \chi_{v_1} + y. \tag{6.12}$$

The Poincaré map between two level sets

$$(\eta_{v_1}, G^{v_1}_0)^{-1}(c, d_0) \quad \text{and} \quad (\eta_{v_1}, G^{v_1}_2)^{-1}(c, d_2)$$

is

$$P(y) = \phi_{\chi^{v_1}}\left(\frac{d_2 - y^{\alpha_2}_{\hat{k}_2}}{(\chi^{v_1})^{\alpha_2}_{\hat{k}_2}}, y\right) \tag{6.13}$$

Notice that the flow of $\chi_{v_1} = X_{\eta_{v_1}}$ preserves η_{v_1}. Clearly, the Poincaré maps can be considered between level set of functions $G^{v_1}_r$, $r = 0, 2$. So we define:

Definition 6.14. For $r = 0, 2$, let $\bar{\mathcal{F}}^{v_1}_r$ be the foliation constituted by the level set of functions $G^{v_1}_r(y) = y^{\alpha_r}_{\hat{k}_r}$.

Instead of two foliations $\mathcal{F}^{v_1}_r$, $r = 0, 2$, we will consider foliations $\bar{\mathcal{F}}^{v_1}_r$. Every leaf of $\bar{\mathcal{F}}^{v_1}_r$ is equipped with a Poisson structure, $\pi^{v_1}_r$, which has η_{v_1} as a Casimir and the level sets of this Casimir are leafs of cosymplectic foliation $\mathcal{F}^{v_1}_r$. Cleary, for a fixed $r \in \{0, 2\}$, the leafs of $\bar{\mathcal{F}}^{v_1}_r$ are Poisson diffeomorphic through Poincaré maps of χ^{v_1}. Since these Poincaré maps are simply translations, see Equation (6.12), these leafs are essentially same (as Poisson manifolds).

Definition 6.15. By $(\Sigma^{v_1}_r, \pi^{v_1}_r)$, $r = 0, 2$ we will denote a typical leaf of Poisson foliation $\bar{\mathcal{F}}^{v_1}_r$.

Notice that $(\Sigma^{v_1}_r, \pi^{v_1}_r)$ is a union of Poisson submanifolds equipped with Dirac bracket.

Proposition 6.16. *If we identify $\Pi_{\gamma,\gamma'}$ with $(\Sigma^{v_1}_0, \pi^{v_1}_0)$ and $L_{\gamma,\gamma'}(\Pi_{\gamma,\gamma'})$ with $(\Sigma^{v_1}_2, \pi^{v_1}_2)$ via Poincaré maps (translations) of type defined in Equation* (6.12) *then $L_{\gamma,\gamma'}$ is Poisson map, see Figure 6.*

Proof. We decompose $L_{\gamma,\gamma'}$ into three Poincaré maps $P^{v_1}_1$, $P^{v_1}_2$ and $P^{v_1}_3$ where $P^{v_1}_1$ is used to identify $\Pi_{\gamma,\gamma'}$ with $(\Sigma^{v_1}_0, \pi^{v_1}_0)$, $P^{v_1}_3$ is used to identify $L_{\gamma,\gamma'}(\Pi_{\gamma,\gamma'})$ with $(\Sigma^{v_1}_2, \pi^{v_1}_2)$ and $P^{v_1}_2$ is the Poincaré map from $(\Sigma^{v_1}_0, \pi^{v_1}_0)$ to $(\Sigma^{v_1}_2, \pi^{v_1}_2)$. P_1 and P_3 are Poisson since they are simple translations. Item (3) of Lemma 6.13 shows that P_2, defined in Equation (6.13), is a Poisson map. □

We describe now the matrix representation of the Dirac bracket structure $\pi^{v_1}_r$, $r = 0, 2$.

Lemma 6.17. *The matrix representation of the Dirac bracket generated in* $\mathrm{int}(\Pi_{v_1})$ *by second class constrains η_{v_1} and $G^{v_1}_r$, $r = 0, 2$, is*

$$(\pi^{v_1}_{\mathrm{Dirac},r})^\sharp = B_{v_1} - C_{(v_1,r)} \ , \tag{6.14}$$

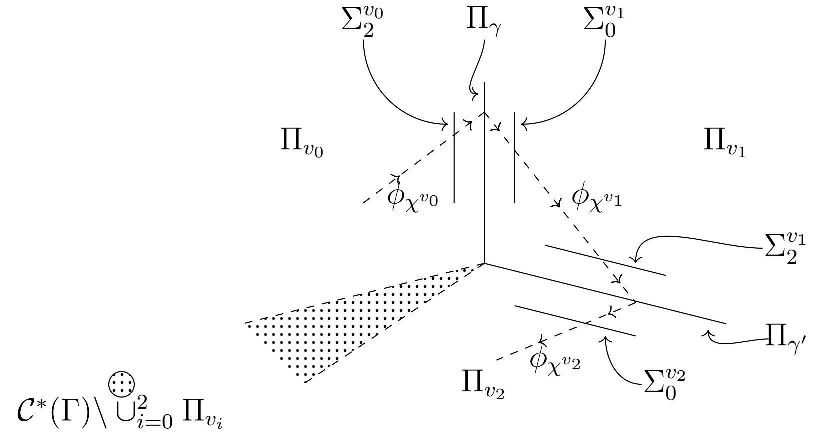

FIGURE 6. Illustration of Proposition 6.16.

where $C_{(v_1,r)} = [C^{\alpha,\beta}_{(v_1,r)}]_{\alpha,\beta}$ *with* $C^{\alpha,\beta}_{v_1,r} = [c_{\hat{i}\hat{j}}(\alpha,\beta,v_1,r)]_{(\hat{i},\hat{j})\in[\alpha]\times[\beta]}$ *where*

$$c_{\hat{i}\hat{j}}(\alpha,\beta,v_1,r) = \frac{1}{(\chi^{v_1})^{\alpha_r}_{\hat{k}_r}}\left((\chi^{v_1})^{\alpha}_{\hat{i}}\, b^{\alpha_r,\beta}_{\hat{k}_r\hat{j}} + b^{\alpha,\alpha_r}_{\hat{i}\hat{k}_r}\,(\chi^{v_1})^{\beta}_{\hat{j}}\right).$$

In the matrix $(\pi^{v_1}_{\mathrm{Dirac},r})^\sharp$ *the line and the column associated to* $G^{v_1}_r = y^{\alpha_r}_{\hat{k}_r}$ *are null. Removing these line and column one obtains the matrix representation of the Poisson structure* $\pi^{v_1}_r$, $r = 0, 2$.

Proof. Notice that

$$(\pi^{v_1}_{\mathrm{Dirac},r})^\sharp = \Big[\, \{y^{\alpha}_{\hat{i}}, y^{\beta}_{\hat{j}}\}\, \Big]_{(\hat{i},\hat{j})\in[\alpha]\times[\beta]}.$$

A simple calculation yields (6.14), see Definition of $\{.,.\}_{\mathrm{Dirac}}$ at Equation (3.4). The rest of the proof is trivial considering that the function $G^{v_1}_r = y^{\alpha_r}_{\hat{k}_r}$ is a Casimir of $\pi^{v_1}_{\mathrm{Dirac},r}$ and the submanifold $\Sigma^{v_1}_r$ is a level set $\Sigma^{v_1}_r$. □

Remark 6.18. The Poisson structure considered on domain of $L_{\gamma,\gamma'}$ is $\pi^{v_1}_0$ and on its image is $\pi^{v_1}_0$, see Proposition 6.16. Furthermore, this Poisson structure only depends on γ and is independent of $L_{\gamma,\gamma'}$ so we can actually consider this structure on whole Π_γ. From now on we will do so.

The main result of this manuscript is

Theorem 6.19. *Let*

$$\xi : v_0 \xrightarrow{\gamma_0} v_1 \xrightarrow{\gamma_1} v_2 \longrightarrow \ldots \longrightarrow v_m \xrightarrow{\gamma_m} v_{m+1} \tag{6.15}$$

be a heteroclinic path. Then

(1) *For every* $l = 1, \ldots, m$, *two Poisson submanifolds,* $(\Sigma_2^{v_{l-1}}, \pi_2^{v_{l-1}})$ *and* $(\Sigma_0^{v_l}, \pi_0^{v_l})$ *(see Definition 6.15), are Poisson diffeomorphic. In other words, the Poisson structures considered on* $\Pi_{\gamma_{l-1},\gamma_l}$

from $\operatorname{int}(\Pi_{v_{l-1}}, B_{v_{l-1}})$ *and* $\operatorname{int}(\Pi_{v_l}, B_{v_l})$ *are the same, see Figure 6.*

(2) If we identify every $\Pi_{\gamma_{l-1},\gamma_l}$, $l = 1, \ldots, m$ *with Poisson submanifold* $(\Sigma_2^{v_{l-1}}, \pi_2^{v_{l-1}})$ *(or equivalently with* $(\Sigma_0^{v_l}, \pi_0^{v_l})$*), the skeleton flow map of* χ *along* ξ *(see Definition 5.14),*

$$\pi_\xi := L_{\gamma_{m-1},\gamma_m} \circ \ldots \circ L_{\gamma_0,\gamma_1} ,$$

is a Poisson map.

Proof. Item (2) if the lemma is a consequence of Item (1) and Proposition 6.16.

Notice that the two sectors $\Pi_{v_{l-1}}$ and Π_{v_l} are only different in the group $\alpha_2^{v_{l-1}} = \alpha_0^{v_l}$, where $y_{\hat{k}_0}^{\alpha_0^{v_1}} = 0$ for the elements of $\Pi_{v_{l-1}}$ and $y_{j_{\alpha_0^{v_1}}}^{\alpha_0^{v_1}} = 0$ for the elements Π_{v_l}, see Remark 6.10. Let

$$P_{v_{l-1},v_l} = \operatorname{diag}(P^1_{v_{l-1},v_l}, \ldots, P^p_{v_{l-1},v_l}) + T_{l-1,l},$$

be the map defined by following items

1) For $\beta \neq \alpha_0^{v_l}$ the associated component $P^\beta_{v_{l-1},v_l}$ is the identity map.
2)

$$(P_{v_{l-1},v_l}(y))_{\hat{i}}^{\alpha_0^{v_l}} = \begin{cases} y_{\hat{i}}^{\alpha_0^{v_l}} - y_{j_{\alpha_0^{v_l}}}^{\alpha_0^{v_l}} & \text{if } \ \hat{i} \neq \hat{k}_0 \\ -y_{j_{\alpha_0^{v_l}}}^{\alpha_0^{v_l}} & \text{if } \ \hat{i} = \hat{k}_0 \end{cases}$$

3) The constant vector $T_{l-1,l}$ is determined such that

$$P_{v_0,v}(\Sigma_2^{v_{l-1}}) = \Sigma_0^{v_l}.$$

Notice that $\Sigma_2^{v_{l-1}}$ is a level set of the function $G_2^{v_{l-1}} = y_{j_{\alpha_0^{v_l}}}^{\alpha_0^{v_l}}$, $\Sigma_0^{v_l}$ is a level set of $G_0^{v_l} = y_{\hat{k}_0}^{\alpha_0^{v_l}}$ and P_{v_{l-1},v_l} sends the level set of the function $G_2^{v_{l-1}}$ to level sets of the function $G_0^{v_l}$. This means that Item (3) above is feasible. Using Item (3) above we set

$$P_{v_{l-1},v_l} : U_{\Sigma_2^{v_{l-1}}} \to U_{\Sigma_0^{v_l}},$$

where $U_{\Sigma_2^{v_{l-1}}}$ and $U_{\Sigma_0^{v_l}}$ are neighborhoods of $\Sigma_2^{v_{l-1}}$ and $U_{\Sigma_0^{v_l}}$, respectively, such that the associated Dirac brackets are defined.

It takes a simple calculation to verify that $(DP_{v_{l-1},v_l})E_{v_{l-1}} = E_{v_l}$. This fact together with the condition (3.3) and the definitions of $B_{v_{l-1}}, B_{v_l}$ (see Lemma 6.7) yields that

$$P_{v_{l-1},v_l} : (U_{\Sigma_2^{v_{l-1}}}, B_{v_{l-1}}) \to (U_{\Sigma_0^{v_l}}, B_{v_l})$$

is a Poisson map, i.e. P_{v_{l-1},v_l} preserves the ambient Poisson structure.

We have

$$
\begin{aligned}
&\eta_{v_l}\circ P_{v_{l-1},v_l}(y) = \\
&\quad \sum_{\beta\neq\alpha_0^{v_l}}\sum_{\hat{j}\in[\beta]}\lambda_\beta q_j^\beta y_j^\beta + \left(\sum_{\hat{i}\neq\hat{k}_0}\lambda_{\alpha_0^{v_l}} q_{\hat{i}}^{\alpha_0^{v_l}}\big(y_i^{\alpha_0^{v_l}} - y_{j_{\alpha_0^{v_l}}}^{\alpha_0^{v_l}}\big)\right) - \lambda_{\alpha_0^{v_l}} q_{\hat{k}_0}^{\alpha_0^{v_l}} y_{j_{\alpha_0^{v_l}}}^{\alpha_0^{v_l}} \\
&\quad = \sum_{\beta\neq\alpha_0^{v_l}}\sum_{\hat{j}\in[\beta]}\lambda_\beta q_j^\beta y_j^\beta + \left(\sum_{\hat{i}\neq\hat{k}_0}\lambda_{\alpha_0^{v_l}} q_{\hat{i}}^{\alpha_0^{v_l}} y_i^{\alpha_0^{v_l}}\right) - \lambda_{\alpha_0^{v_l}}\left(\sum_{\hat{i}} q_{\hat{i}}^{\alpha_0^{v_l}}\right) y_{j_{\alpha_0^{v_l}}}^{\alpha_0^{v_l}} \\
&\quad = \sum_{\beta\neq\alpha_0^{v_l}}\sum_{\hat{j}\in[\beta]}\lambda_\beta q_j^\beta y_j^\beta + \left(\sum_{\hat{i}}\lambda_{\alpha_0^{v_l}} q_{\hat{i}}^{\alpha_0^{v_l}} y_i^{\alpha_0^{v_l}}\right) - 1 = \eta_{v_{l-1}} - 1,
\end{aligned}
\tag{6.16}
$$

where we used the fact that $-\sum_{\hat{i}} q_{\hat{i}}^{\alpha_0^{v_l}} = q_{j_{\alpha_0^{v_l}}}^{\alpha_0^{v_l}} - 1$. In other word, P_{v_{l-1},v_l} sends the second class constraint of $\eta_{v_{l-1}}$ of $\Sigma_2^{v_{l-1}}$ to second class constraint η_{v_l} of $\Sigma_0^{v_l}$ (ignoring addition of -1 which does not do any harm). Furthermore,

$$G_2^{v_l}\circ P_{v_{l-1},v_l} = -G_0^{v_l} + T_{l-1,l}\ .$$

Notice that, as mentioned in Remark 3.9, the Poisson structure $\pi_2^{v_{l-1}}$ is independent of the choice of second class constraint $G_0^{v_l}$. Since $-G_0^{v_l}$ is a second class constraint as well, considering it as the second class constraint , beside $\eta_{v_{l-1}}$, as the constraint used to define $\pi_2^{v_{l-1}}$, we have shown P_{v_{l-1},v_l} sends the second class constraints of $\Sigma_2^{v_{l-1}}$ to second class constraints of $\Sigma_0^{v_l}$. Consequently, the map

$$(P_{v_{l-1},v_l})|_{\Sigma_2^{v_{l-1}}} : (\Sigma_2^{v_{l-1}},\pi_2^{v_{l-1}}) \to (\Sigma_0^{v_l},\pi_0^{v_l}),$$

is a Poisson map since P_{v_{l-1},v_l} preserves the ambient Poisson structure as well.

We denote by $P_2^{v_{l-1}}$ the map used to identify $\Pi_{\gamma_{l-1},\gamma_l}$ with $(\Sigma_2^{v_{l-1}},\pi_2^{v_{l-1}})$ via flow of $\chi^{v_{l-1}}$, see Equation (6.12), and by $P_0^{v_l}$ the map used to identify $\Pi_{\gamma_{l-1},\gamma_l}$ with $(\Sigma_0^{v_l},\pi_0^{v_l})$ via flow of χ^{v_l}. Another consequence of Equation 6.16 is that P_{v_{l-1},v_l} sends $\chi^{v_{l-1}}$ to χ^{v_l} which means that the following diagram

$$
\begin{tikzcd}
 & \Pi_{\gamma_{l-1},\gamma_l} \arrow[dr, "P_0^{v_l}"] & \\
\Sigma_2^{v_{l-1}} \arrow[ur, "P_2^{v_{l-1}}"] \arrow[rr, "P_{v_{l-1},v_l}"] & & \Sigma_0^{v_l}
\end{tikzcd}
$$

is commutative. This finishes the proof. $\square$

The Poisson structures defined on the sectors associated to an edge depend only on the edge and on the skeleton vector field, and is independent of the the heteroclinic path containing the edge (see Remark 6.18). Let us denote this structure by $\{\cdot,\cdot\}_\gamma$. Our result holds for the skeleton flow map (see Definition 5.17) as an immediate consequence of Theorem 6.19.

Theorem 6.20. *Let $\mathscr{B}_S(\chi)$ denote the set of all S-branches of the skeleton vector field ξ (see Definition 5.16) and set $D_S := \cup_{\xi\in\mathscr{B}_S(\chi)}\Pi_\xi$ to be the open submanifold of*

$$(\Pi_S, \{\cdot,\cdot\}_S) := \cup_{\gamma\in S}(\Pi_\gamma, \{\cdot,\cdot\}_\gamma),$$

with the same Poisson structure. Then the skeleton flow map $\pi_S : (D_S, \{\cdot,\cdot\}_S) \to (\Pi_S, \{\cdot,\cdot\}_S))$ is Poisson.

7. Example

We will now present an example of a Hamiltonian polymatrix replicator system with a non trivial dimension. This example was chosen to provide an illustration of the concepts and main results of this paper. In particular it has a small structural set with a simple heteroclinic network.

7.1. **The fish example.** Consider the polymatrix replicator system defined by matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

We denote by X_A the vector field associated to this polymatrix replicator that is defined on the polytope

$$\Gamma_{(5,2)} := \Delta^4 \times \Delta^1 .$$

The point

$$q = \left(\frac{1}{9}, \frac{1}{3}, \frac{1}{9}, \frac{1}{3}, \frac{1}{9}, \frac{2}{3}, \frac{1}{3}\right) \in \Gamma_{(5,2)}$$

satisfies

(1) $Aq = (0,0,0,0,0,0,0)$;
(2) $q_1 + q_2 + q_3 + q_4 + q_5 = 1$ and $q_6 + q_7 = 1$,

where q_i stands for i-th component of the vector q, and hence is an equilibrium of X_A (see Proposition 4.3). Since matrix A is skew-symmetric, the associated polymatrix replicator is conservative (see Definition 4.4).

The polytope $\Gamma_{(5,2)}$ has seven faces labelled by an index j ranging from 1 to 7, and designated by $\sigma_1, \ldots, \sigma_7$. The vertices of the phase space $\Gamma_{(5,2)}$ are also labelled by $i \in \{1, \ldots, 10\}$, and designated by $v_1, \ldots, v_{10}$, as described in Table 1.

Vertex	$\Gamma_{(5,2)}$
$v_1 = (1,6)$	$(1,0,0,0,0,1,0)$
$v_2 = (1,7)$	$(1,0,0,0,0,0,1)$
$v_3 = (2,6)$	$(0,1,0,0,0,1,0)$
$v_4 = (2,7)$	$(0,1,0,0,0,0,1)$
$v_5 = (3,6)$	$(0,0,1,0,0,1,0)$

Vertex	$\Gamma_{(5,2)}$
$v_6 = (3,7)$	$(0,0,1,0,0,0,1)$
$v_7 = (4,6)$	$(0,0,0,1,0,1,0)$
$v_8 = (4,7)$	$(0,0,0,1,0,0,1)$
$v_9 = (5,6)$	$(0,0,0,0,1,1,0)$
$v_{10} = (5,7)$	$(0,0,0,0,1,0,1)$

Table 1. Identification of the ten vertices of the polytope, $v_1, \ldots, v_{10}$ in $\Gamma_{(5,2)}$.

The skeleton character χ_A of X_A is displayed in Table 2. (See Definition 5.3 and Proposition 5.5.)

χ^v_σ	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
v_1	$*$	1	0	0	0	$*$	-1
v_2	$*$	0	-1	-1	-2	1	$*$
v_3	-1	$*$	1	0	0	$*$	0
v_4	0	$*$	1	0	-1	0	$*$
v_5	0	-1	$*$	1	0	$*$	0
v_6	1	-1	$*$	1	-1	0	$*$
v_7	0	0	-1	$*$	1	$*$	0
v_8	1	0	-1	$*$	0	0	$*$
v_9	0	0	0	-1	$*$	$*$	1
v_{10}	2	1	1	0	$*$	-1	$*$

Table 2. The skeleton character χ_A of X_A, where the symbol $*$ in the i-th line and j-th column of the table means that the vertex v_i does not belong to the face σ_j of the polytope $\Gamma_{(5,2)}$.

The edges of $\Gamma_{(5,2)}$ are designated by $\gamma_1, \ldots, \gamma_{25}$, according to Table 3, where we write $\gamma = (i,j)$ to mean that γ is an edge connecting the vertices v_i and v_j. This model has 25 edges: 12 neutral edges,

$$\gamma_2, \gamma_3, \gamma_4, \gamma_7, \gamma_8, \gamma_{10}, \gamma_{12}, \gamma_{16}, \gamma_{17}, \gamma_{18}, \gamma_{16}, \gamma_{22}, \gamma_{25},$$

and 13 flowing-edges,

$$\gamma_1, \gamma_5, \gamma_6 \gamma_9, \gamma_{11}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \gamma_{19}, \gamma_{20}, \gamma_{21}, \gamma_{23}, \gamma_{24}.$$

The flowing-edge directed graph of χ_A is depicted in Figure 7.

$\gamma_1 = (1, 2)$	$\gamma_6 = (3, 1)$	$\gamma_{11} = (2, 8)$	$\gamma_{16} = (3, 7)$	$\gamma_{21} = (8, 6)$
$\gamma_2 = (3, 4)$	$\gamma_7 = (2, 4)$	$\gamma_{12} = (1, 9)$	$\gamma_{17} = (4, 8)$	$\gamma_{22} = (5, 9)$
$\gamma_3 = (5, 6)$	$\gamma_8 = (1, 5)$	$\gamma_{13} = (2, 10)$	$\gamma_{18} = (3, 9)$	$\gamma_{23} = (6, 10)$
$\gamma_4 = (7, 8)$	$\gamma_9 = (2, 6)$	$\gamma_{14} = (5, 3)$	$\gamma_{19} = (4, 10)$	$\gamma_{24} = (9, 7)$
$\gamma_5 = (10, 9)$	$\gamma_{10} = (1, 7)$	$\gamma_{15} = (6, 4)$	$\gamma_{20} = (7, 5)$	$\gamma_{25} = (8, 10)$

TABLE 3. Edge labels.

From this graph we can see that

$$S = \{\, \gamma_1 = (1, 2) \,\}$$

is a structural set for χ_A (see Definition 5.16) whose S-branches denoted by $\xi_1, \ldots, \xi_5$ are displayed in Table 4, where we write $\xi_i = (j, k, l, \ldots)$ to indicate that ξ_i is a path from vertex v_j passing along vertices $v_k, v_l, \ldots$.

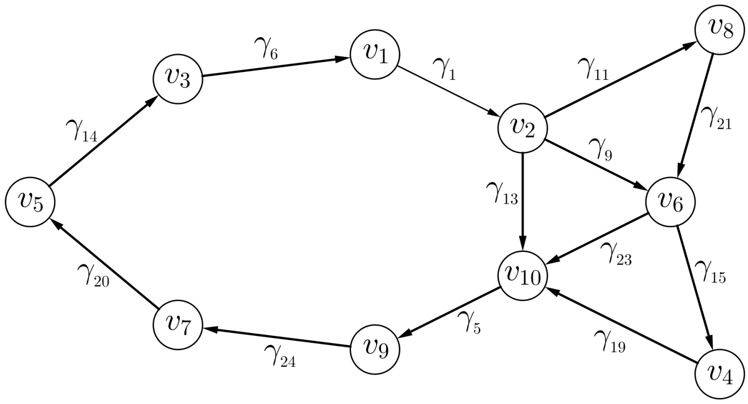

FIGURE 7. The oriented graph of χ_A.

From\To	$\gamma_1 = (1, 2))$
$\gamma_1 = (1, 2)$	$\xi_1 = (1, 2, 10, 9, 7, 5, 3, 1, 2)$
	$\xi_2 = (1, 2, 6, 10, 9, 7, 5, 3, 1, 2)$
	$\xi_3 = (1, 2, 6, 4, 10, 9, 7, 5, 3, 1, 2)$
	$\xi_4 = (1, 2, 8, 6, 10, 9, 7, 5, 3, 1, 2)$
	$\xi_5 = (1, 2, 8, 6, 4, 10, 9, 7, 5, 3, 1, 2)$

TABLE 4. S-branches of χ_A.

Considering the vertex v_1, which has the incoming edge $v_3 \xrightarrow{\gamma_6} v_1$ and the outgoing edge $v_1 \xrightarrow{\gamma_1} v_2$, we will now illustrate Proposition 6.16.

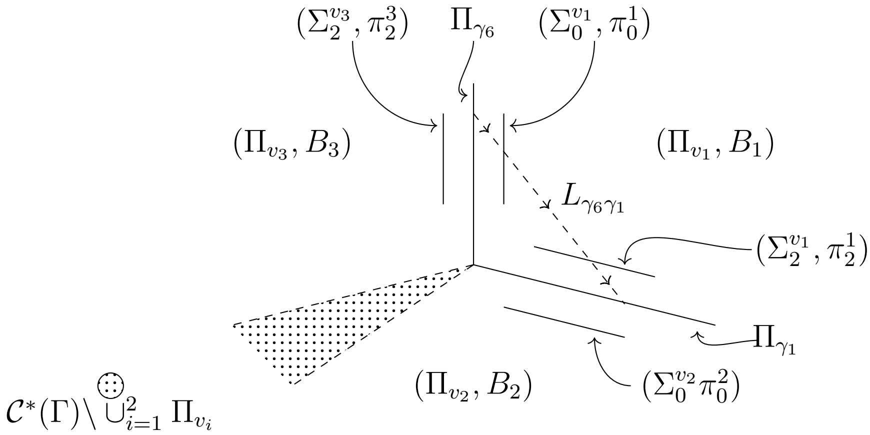

FIGURE 8. Illustration of Proposition 6.16 for the example.

By straightforward calculations we obtain

$$B_1 = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ -2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 & 2 \\ -1 & -1 & -1 & -2 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 2 & 1 & 1 & -1 \\ -2 & 0 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 & -2 \\ 1 & 1 & 1 & 2 & 0 \end{pmatrix}$$

and

$$B_3 = \begin{pmatrix} 0 & -2 & -1 & -1 & -1 \\ 2 & 0 & 2 & 1 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix},$$

and by (6.14) we get

$$(\pi^{v_1}_{\mathrm{Dirac},2})^\sharp = (\pi^{v_2}_{\mathrm{Dirac},0})^\sharp = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$(\pi^{v_3}_{\mathrm{Dirac},2})^\sharp = (\pi^{v_0}_{\mathrm{Dirac},1})^\sharp = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

The matrix $(\pi^{v_2}_{\mathrm{Dirac},0})^\sharp$ is the representation of the Poisson structure on Π_{γ_6} in the coordinates $(y_2, y_3, y_4, y_5, y_7)$. Notice that $y_2 = 0$ on Π_{γ_6}. Similarly, the matrix $(\pi^{v_0}_{\mathrm{Dirac},1})^\sharp$ is the representation of the Poisson structure on Π_{γ_1} in the same coordinates $(y_2, y_3, y_4, y_5, y_7)$. Notice again that $y_7 = 0$ on Π_{γ_1}. Now the matrix representation of $L_{\gamma_6\gamma_1}$ in the coordinates $(y_2, y_3, y_4, y_5, y_7)$ is

$$L_{\gamma_6\gamma_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

A simple calculation shows that

$$L_{\gamma_6\gamma_1}\,(\pi^{v_2}_{\mathrm{Dirac},0})^{\sharp}\,(L_{\gamma_6\gamma_1})^t = (\pi^{v_0}_{\mathrm{Dirac},1})^{\sharp},$$

confirming the fact that the asymptotic Poincaré map $L_{\gamma_6\gamma_1}$ is Poisson (see(3.3) in Definition 3.1).

Consider now the subspaces of $\mathbb{R}^7$

$$H = \left\{ (x_1, \ldots, x_7) \in \mathbb{R}^7 \ : \ \sum_{i=1}^{5} x_i = 1, \ \sum_{i=6}^{7} x_i = 1 \right\}$$

and

$$H_0 = \left\{ (x_1, \ldots, x_7) \in \mathbb{R}^7 \ : \ \sum_{i=1}^{5} x_i = 0, \ \sum_{i=6}^{7} x_i = 0 \right\}.$$

For the given matrix A, its null space $\mathrm{Ker}(A)$ has dimension 3. Take a non-zero vector $w \in \mathrm{Ker}(A) \cap H_0$. For example,

$$w = (-2, 3, -2, 3, -2, -3, 3)\,.$$

The set of equilibria of the natural extension of X_A to the affine hyperplane H is

$$\mathrm{Eq}(X_A) = \mathrm{Ker}(A) \cap H = \{q + tw \colon t \in \mathbb{R}\}\,.$$

The Hamiltonian of X_A is the function $h_q : \Gamma_{(5,2)} \to \mathbb{R}$

$$h_q(x) := \sum_{i=1}^{7} q_i \log x_i\,,$$

where q_i is the i-th component of the equilibrium point q (see Theorem 4.5). Another integral of motion of X_A is the function $h_w : \Gamma_{(5,2)} \to \mathbb{R}$

$$h_w(x) := \sum_{i=1}^{7} w_i \log x_i\,,$$

where w_i is the i-th component of w, which is a Casimir of the underlying Poisson structure.

The skeletons of h_q and h_w are respectively $\eta_q, \eta_w : \mathcal{C}^*(\Gamma_{(5,2)}) \to \mathbb{R}$,

$$\eta_q(y) := \sum_{i=1}^{7} q_i y_i \quad \text{and} \quad \eta_w(y) := \sum_{i=1}^{7} w_i y_i\,,$$

(see Proposition 6.1), which we use to define $\eta : \mathcal{C}^*(\Gamma_{(5,2)}) \to \mathbb{R}^2$,

$$\eta(y) := (\eta_q(y), \eta_w(y)).$$

Consider the skeleton flow map $\pi_S : \Pi_S \to \Pi_S$ of χ_A (see Definition 5.17). Notice that $\Pi_S = \Pi_{\gamma_1}$, where by Proposition 5.18, $\Pi_{\gamma_1} = \bigcup_{i=1}^5 \Pi_{\xi_i}$ (mod 0). By Proposition 6.1 the function η is invariant under π_S. Moreover, the skeleton flow map π_S is Hamiltonian w.r.t. a Poisson structure on the system of cross sections Π_S (see Theorem 6.19).

For all $i = 1, \ldots, 5$, the polyhedral cone Π_{ξ_i} has dimension 4. Hence, each polytope $\Delta_{\xi_i,c} := \Pi_{\xi_i} \cap \eta^{-1}(c)$ is a 2-dimensional polygon.

Remark 7.1. We came from dimension 5 to 2. This will happen for any other conservative polymatrix replicator with the same number of groups and the same number of strategies per group. In fact when $n - p$ is odd, where n is the total number of strategies in the population and p is the number of groups, we will have a minimum drop of 3 dimension. The reason is that a Poisson manifold with odd dimension (in this example is 5) has at least one Casimir, and considering the transversal section we drop two dimensions from the symplectic part (not from the Casimir). So in total we drop a minimum of three dimensions. If the original Poisson structure has more Casimirs, the invariant submanifolds yielded geometrically, are going to have even less dimensions, which is good as long as it not zero. In the case of an even dimension, the drop will be at least of two dimensions.

By invariance of η, the set $\Delta_{S,c}$ is also invariant under π_S. Consider now the restriction $\pi_{S|\Delta_{S,c}}$ of π_S to $\Delta_{S,c}$. This is a piecewise affine area preserving map. Figure 9 shows the domain $\Delta_{S,c}$ and 20 000 iterates by π_S of a point in $\Delta_{S,c}$. Following the itinerary of a random point we have picked the following heteroclinic cycle consisting of 4 S-branches

$$\xi := (\xi_4, \xi_1, \xi_3, \xi_4)\,.$$

The map π_ξ is represented by the matrix

$$M_\xi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -\frac{13}{2} & 2 & -\frac{3}{2} & 0 \\ 1 & 0 & 1 & -1 & 1 & 2 & 0 \\ -1 & 2 & -1 & \frac{15}{2} & -2 & \frac{5}{2} & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of M_ξ, besides 0 and 1 (with geometric multiplicity 3 and 2, respectively), are

$$\lambda_u = 5.31174..., \quad \text{and} \quad \lambda_s = \lambda_u^{-1}.$$

Remark 7.2. The determinant of $(\pi^{v_2}_{\text{Dirac},0})^\sharp$ is zero which means that the Poisson structure on Π_{γ_6} is non-degenerate. So, Π_6 has a two dimensional symplectic foliation invariant under the asymptotic Poincaré map. The leaf of this foliation are affine spaces parallel to the kernel of

$$(\pi^{v_2}_{\text{Dirac},0})^\sharp|_{\Pi_{\gamma_6}} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix},$$

i.e. the set of the form

$$\{(q_3, q_4, q_5, q_7) + (s, t, -t, -s) \mid (q_3, q_4, q_5, q_7) \in \Pi_{\gamma_6}\ s, t \in \mathbb{R}\} \cap \Pi_{\gamma_6}.$$

The restriction of the asymptotic Poincaré map to these leafs is a symplectic map. One important consequence is that its eigenvalues are of the form λ and $\frac{1}{\lambda}$.

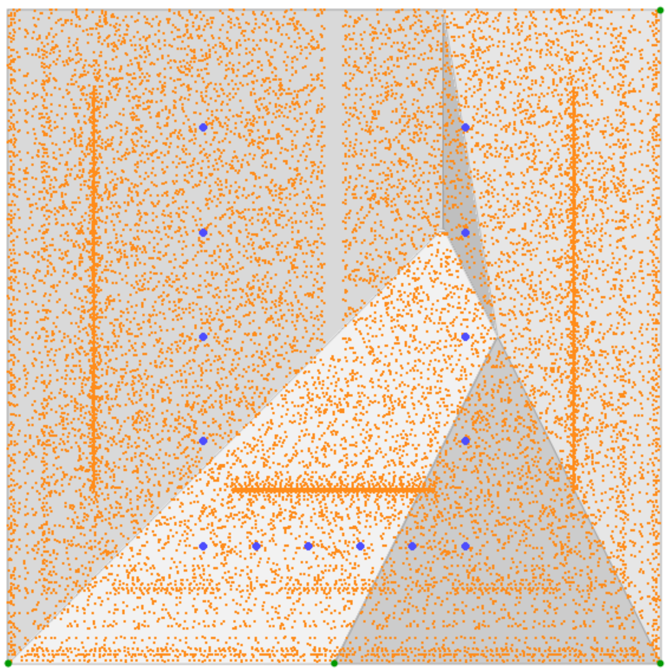

FIGURE 9. Plot of 20 000 iterates (in orange) by π_S of a point in $\Delta_{S,c}$, with $c = \left(\frac{1}{3}, -0.5\right)$, the iterates of the periodic point $\mathbf{p_0}$ (in green) of the skeleton flow map π_S with period 4, and the iterates of another periodic point of the skeleton flow map π_S with period 14 (in blue).

An eigenvector associated to the eigenvalue 1 is

$$\mathbf{p_0} = (0., 0.5, 1., 0., 0., 0., 0.)\,.$$

We have chosen $c := (c_1, c_2) = \left(\frac{1}{3}, -0.5\right)$ so that $\eta(\mathbf{p_0}) = c$, *i.e.*, $\mathbf{p_0} \in \Delta_{S,c}$. In fact we have $\mathbf{p_0} \in \Delta_{\xi_1,c} \subset \Delta_{\gamma_1,c}$. Hence $\mathbf{p_0}$ is a periodic point of the skeleton flow map π_S with period 4 (whose iterates are represented by the green dots in Figure 9).

Figure 9 also depicts the polygons $\Delta_{\xi_1,c}, \Delta_{\xi_2,c}, \Delta_{\xi_3,c}, \Delta_{\xi_4,c}, \Delta_{\xi_5,c}$ contained in Δ_{γ_1}, and the orbit of another periodic point of the skeleton flow map π_S with period 14 (represented by the blue dots in Figure 9).

By [3, Theorem 8.7] we can deduce the existence of chaotic behaviour for the flow of X_A in some level set $h_q^{-1}(c_1/\epsilon) \cap h_w^{-1}(c_2/\epsilon)$, with $c = (c_1, c_2)$ chosen above and for all small enough $\epsilon > 0$.

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