

A Note on Numbers

January 18, 2021

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Abstract

When are all positions of a game numbers? We show that two properties are necessary and sufficient. These properties are consequences of that, in a number, it is not an advantage to be the first player. One of these properties implies the other. However, checking for one or the other, rather than just one, can often be accomplished by only looking at the positions on the ‘board’. If the stronger property holds for all positions, then the values are integers.

KEYWORDS: Combinatorial Game Theory, numbers, BLUE-RED-HACKENBUSH, DOMINO SHAVE, SHOVE, PUSH, LENRES, POLYCHROMATIC CHOMP, PARTIZAN TURNING TURTLES, DIVISORS, BLUE-RED-CHERRIES, CUTCAKE, EROSION.

1 Introduction

When analyzing games, an early question is: is it possible that all the positions are numbers? If that is true, then it is easy to determine the outcome of a disjunctive sum of positions, just add up the numbers. It is also easy to find the best move, just play the summand with the largest denominator. The problem is how to recognize when all the positions are numbers.

Siegel [8], page 81, exercise 3.15, states “If every incentive of G is negative then G is a number”. This does not provide much insight or intuition. In fact, in most non-all-small-games, there are non-zero positions, some of which are numbers and others not. Let \mathbf{S} be a set of positions of a ruleset. It is called a *hereditary closed set of positions of a ruleset* (HCR) if it is closed under taking options. These HCR sets are the natural objects to consider.