

Ordinal Sums, CLOCKWISE HACKENBUSH, and DOMINO SHAVE

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Dedicated to Elwyn R. Berlekamp, John H. Conway and Richard K. Guy, they
taught us so much.

Abstract

We present two rulesets, DOMINO SHAVE and CLOCKWISE HACKENBUSH. The first is somehow natural and, as special cases, includes STIRLING SHAVE and Hetyei’s Bernoulli game. CLOCKWISE HACKENBUSH seems artificial yet it is equivalent to DOMINO SHAVE. From the pictorial form of the game, and a knowledge of HACKENBUSH, the decomposition into ordinal sums is immediate. The values of CLOCKWISE BLUE-RED HACKENBUSH are numbers and we provide an explicit formula for the ordinal sum of numbers where the literal form of the base is $\{x|\}$ or $\{|\ x\}$, and x is a number. That formula generalizes van Roode’s signed binary number method for BLUE-RED HACKENBUSH.

KEYWORDS: Combinatorial Game Theory, HACKENBUSH, van Roode’s method, ordinal sum.

1 Introduction

HACKENBUSH is a central game in *Winning Ways* [4]. It has many interesting properties. One that will be central to this paper is the relationship between the ordinal sum decomposition and the valuation scheme for paths and trees. The literature also includes variants with new intriguing properties in new contexts. For example, YELLOW-BROWN HACKENBUSH [3] and all-small games;