Preventing environmental disasters in investment under uncertainty

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Abstract

The paper considers a firm that has the option to invest in a project with an unknown profitability, which is affected by general market uncertainty. The project has the adverse effect that it can cause environmental damage. In case the firm has the option to undertake preventive investment at the time of market entry, we get that preventive investment is significant when (i) the project revenue is large, (ii) the environmental incidents potentially cause a huge reduction of firm value, and (iii) when preventive investment substantially decreases the probability of environmental damage occurrence. The optimality of such a preventive investment results in a significant delay of the project investment. When the firm has the possibility to invest in the project first and do the preventive investment later, this will accelerate the project investment and will result in a larger preventive investment when it indeed will decide to do that one later.

Keywords: preventive investment; real options; environmental risk.

1 Introduction

Environmental awareness has prompted many governments to introduce regulations that stimulate cleaner and safer ways of production. Despite these measures, a large number of environmental incidents occurs regularly as a result of industrial development posing a threat to environment and local communities. These include, for example, air pollution, chemical and oil spills, water contamination etc. In many cases, however, major pollution incidents can be prevented if appropriate measures are in place. Typically, the companies involved in such incidents are not directly liable for their consequences, which may discourage them from taking appropriate prevention measures. On the other hand, there exists empirical evidence that the occurrence of environmental incidents has a negative impact on the firm's value. For example, Lundgren and Olsson (2010) find a significant negative value loss for European companies that incurred incidents in violation of international environmental norms. Furthermore, these effects may reach beyond the violating company itself. In an example from the automotive industry, Jacobs and Singhal (2020) find a statistically significant negative effect of the September 2015 Volkswagen emissions scandal on the stock prices of its suppliers and customers. The negative stock price reaction creates monetary incentives for the companies to increase expenditures to mitigate environmental risks, which will in turn affect the incentive to invest in polluting projects in the first place. Several early contributions in environmental finance suggest that better environmental management practices, such as investments in abatement capital, reduce the firms' systematic risk and, thus, positively affect the stock prices (Hamilton, 1995; Feldman *et al.*, 1997).

In this article, we study the optimal timing of investment decisions in projects with a potential for environmental hazard. In our model, the firm can, to some extent, control the amount of environmental risk they are willing to bear. In particular, the firms may undertake a voluntary preventive investment to reduce the intensity of environmental incidents. We investigate the incentives of polluting firms to introduce preventive measures at the moment of investment, as well as the impact of these measures on the initial decision to undertake such projects.

Among the recent examples of well-publicized environmental disasters is an oil spill from a power plant owned by Nornickel in Russia, in May 2020.[1] As a result of this incident, 21 000 tonnes of oil have contaminated surrounding rivers with the risk to a further spread into the Arctic Ocean, causing an immediate stock price decline of 7.5%[2]. The company was subsequently blamed for operating ageing reservoirs which did not undergo proper refurbishments and, thus, using a dangerous facility in an irresponsible way. Another recent example in mining is the Brumadinho dam disaster in Brazil, in January 2019. The dam owned by iron ore mining company Vale released a mudflow that damaged

[1] https://www.theguardian.com/environment/2020/jun/09/russian-mining-firm-accused-of-using-global-heating-to-avoid-blame-for-oil-spill

[2] https://meduza.io/en/feature/2020/06/08/the-situation-is-dire.

surrounding buildings, agricultural areas and local ecosystem, as well as numerous deaths. In the aftermath of the disaster, Vale market value experienced a 14% decline.[3]

There exists a large body of research that investigates the economic consequences of rare events on investment decisions (Martzoukos and Trigeorgis, 2002). The uncertain arrival of such events is typically modelled as a Poisson process and includes a wide range of applications. Several studies focus on the impact of positive jumps as a result of technological innovation (Farzin *et al.*, 1998; Huisman, 2001; Hagspiel *et al.*, 2020). Another example is policy uncertainty related to either retraction or introduction of favorable regulations, which can result in both positive and negative jumps (Lundgren and Olsson, 2010; Chronopoulos *et al.*, 2016; Dalby *et al.*, 2018). Among the early contributions that specifically consider the impact of catastrophic events on investment is Yin and Newman (1996) that investigates the application to forestry. Farrow and Hayakawa (2002) analyzes a real options problem of a safety investment in the presence of catastrophic incidents risk from the regulator's perspective. Dolan *et al.* (2018) presents a real options valuation model for the mining industry accounting for extreme climate events such as floods and droughts. However, none of the models so far assume that the firms can actively affect the probability of catastrophic events. More specifically, the likelihood of the rare events is considered constant and exogenous to the firm. Our contribution to this stream of literature is that the arrival rate of environmental incidents can be to some extent controlled by the firm as it is able to choose the optimal size of preventive investment.

Our study is also related to the literature on pollution abatement investment (Beavis and Dobbs, 1986; Hartl, 1992). The focus of this early literature is, however, primarily on the compliance with environmental regulations and the incentives to over-comply are not considered. Several papers attempt to explain the reasons behind voluntary abatement investment by consumer preferences, competition and the specifics of the regulatory framework (Arora and Gangopadhyay, 1995; Lundgren, 2003; Maxwell and Decker, 2006). The closest contribution to our model is Lundgren (2003) that studies voluntary investment in abatement capital under uncertainty. The uncertain factors he considers are future evolution of green goodwill, competition and the threat of regulation, modelled by a Poisson jump process. He finds that incentives to undertake voluntary abatement investment are positively related to regulation intensity and competitors' investment, and negatively related to green goodwill uncertainty. Unlike our paper, most of this literature, however, disregards the direct incentive to reduce the risk of environmental disasters by investing in environmentally friendly technologies. In our model, the reason for over-investment in mitigation of environmental risk comes from the ability of the firm to decrease the arrival rate of incidents by optimally timing and sizing preventive investment.

In this regard, the paper is also related to the growing literature on capacity optimization in a real options framework. Traditionally, a real options model determines the optimal investment timing for a

[3]https://www.economist.com/business/2019/03/09/vale-and-the-aftermath-of-a-devastating-dam-failure

given investment size (Dixit and Pindyck, 1994). The main conclusion here is that in more uncertain environments the firms have incentives to delay their investment. More recent contributions allow the firms to choose the optimal size of the investment in addition to timing (Dangl, 1999; Huisman and Kort, 2015; Huberts *et al.*, 2015). The general conclusion of this stream of literature is that uncertainty increases not only the investment timing, but also the investment size. In these models, the channel through which capacity choice affects the profitability of firms' investment is demand. In our model the channel is the risk of environmental incidents. Thus, we contribute to this literature by providing a link between the optimal size of preventive investment and the possibility to increase the project profitability by reducing the environmental risks.

The paper is organized as follows. Section 2 presents and analyzes our model of preventive investment. Comparative statics results are given in Section 3. Section 4 extends the setting of Section 2 by allowing the firm to invest in prevention later than just at the same time of market entry. Section 5 concludes.

2 Baseline model of preventive investment

Consider a firm that faces an opportunity to invest in a project with uncertain profitability by undertaking an investment I_0. Apart from environmental risk, the value of this project is subject to market uncertainty. In particular, the project value is stochastic and evolves according to a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dZ_t, \tag{1}$$

where dZ_t is the increment of a Wiener process, whereas μ and σ represent the drift and volatility parameters, respectively.

In addition to market risk, the firm is also exposed to environmental accident risk. Such accidents represent technical failures that have a serious negative impact on the environment. The regulator requires the firm to make a mandatory preventive investment of minimal size R_0 to reduce the environmental risk when undertaking the project. With the minimal preventive measures, the environmental incidents occur according to a Poisson process $N = \{N_t, t > 0\}$ with intensity λ_0 (failure rate), where N_t denotes the number of incidents that have occurred until time t. Every time an environmental incident occurs, the firm value experiences a decline. We let U_i denote the (random) percentage reduction in the revenue due to the ith incident, with $\{U_i, i \in N\}$ being a sequence of independent and identically distributed random variables with $\mathbb{E}(U_i) = u$, independent of the Poisson process and of the geometric Brownian motion.

In order to reduce the impact of incidents, the firm has the possibility to undertake a voluntary preventive investment of size R to reduce environmental hazard in addition to the minimal investment amount required by the regulators. More specifically, such an investment decreases the arrival rate of environmental incidents to $\lambda(R)$, with $\lambda'(R) < 0$ and $\lambda''(R) > 0$.

In order to maximize the project value, the firm intends to find both the optimal time τ^* and the optimal preventive investment size R^* that would decrease the likelihood of having incidents. Then the firm is facing the following control problem:

$$V(x) = \sup_{\tau,R} \mathbb{E}_x \left[\int_\tau^\infty X_t \prod_{i=1}^{N_{t-\tau}} (1 - U_i) \mathrm{e}^{-rt} dt - (I_0 + R_0 + R)\mathrm{e}^{-r\tau} \right], \tag{2}$$

where I_0 denotes the project investment, R_0 is the mandatory preventive investment, R is the voluntary preventive investment that is additional to R_0, and r denotes the discount rate.

In our baseline model, we assume that both investment decisions must be undertaken at the same time. This can occur, for example, in order to comply with environmental standards, due to the regulation measuring environmental safety. Later, we relax this assumption and let the firm decide upon the optimal timing of both investments.

Note that the problem in (2) can be rewritten as follows

$$V(x) \quad = \quad \sup_{\tau} \mathbb{E}_x \left[\mathrm{e}^{-r\tau} \mathbb{E}_{X_\tau} \left[\int_0^\infty X_t \mathbb{E} \left[\prod_{i=1}^{N_t} (1 - U_i) \right] \mathrm{e}^{-rt} dt \right] - (I_0 + R_0 + R)\mathrm{e}^{-r\tau} \right], \tag{3}$$

where

$$\mathbb{E} \left[\prod_{i=1}^{N_t} (1 - U_i) \right] = \mathbb{E} \left[(\mathbb{E}(1 - U))^{N_t} \middle| N_t \right] = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\lambda(R)t} (\lambda(R)t)^n}{n!} (1 - u)^n = \mathrm{e}^{-\lambda(R)ut} \tag{4}$$

Further, note that

$$\mathbb{E}_x \left[\int_0^\infty \left(X_t \mathrm{e}^{-\lambda(R)ut} - c \right) \mathrm{e}^{-rt} dt \right] = \frac{x}{r - \mu + \lambda(R)u}, \tag{5}$$

Therefore, our problem becomes

$$\sup_{\tau,R} \mathbb{E}_x \left[\mathrm{e}^{-r\tau} \left(\frac{X_\tau}{r - \mu + \lambda(R)u} - I_0 - R_0 - R \right) \right]. \tag{6}$$

Without optimization with respect to R, this becomes a standard investment problem, for which the expected total payoff of the investment undertaken for a given level of x and investment size R, is given by:

$$v(x; R) = \frac{x}{r - \mu + \lambda(R)u} - I_0 - R_0 - R, \tag{7}$$

Given that $\lambda'(R) < 0$, it is evident that the increase in investment amount, R, has an ambiguous effect on the project value $v(x; R)$. On the one hand, mitigation of environmental hazard becomes more costly, meaning that the firm requires a larger profitability to justify the investment. On the other hand the probability of an environmental incident is decreased, making the investment opportunity more profitable, so that the firm is more eager to invest. We can show that the latter effect is dominating for low values of R and the former for large values, so that there exists a unique optimal preventive investment amount.

Let $R^*(x)$ be such that

$$R^*(x) = \operatorname*{argmax}_R \left(\frac{x}{r - \mu + \lambda(R)u} - I_0 - R_0 - R \right), \tag{8}$$

so that $R^*(x)$ is the value of R that maximizes the expected return of the investment, given that the investment decision is taken when the project profitability is equal to x). Then, the solution of problem (6) is given in Proposition 1.

Proposition 1 *For the optimization problem* (6)*, the corresponding value function is given by*

$$V_R(x) = \begin{cases} Ax^\beta & \text{if } x \le x^*, \\ v(x; R^*(x)) & \text{if } x > x^*, \end{cases} \tag{9}$$

where β is equal to

$$\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \tag{10}$$

and the parameters A and x^* are implicitly defined by

$$\begin{cases} \dfrac{x^*}{r - \mu + \lambda(R^*(x^*))u} \left(1 - \dfrac{1}{\beta}\right) - I_0 - R_0 - R^*(x^*) = 0, \\ A = \dfrac{1}{\beta (x^*)^{\beta - 1} (r - \mu + \lambda(R^*(x^*))u)}. \end{cases} \tag{11}$$

In order to proceed with the calculations, we need to specify a particular functional form for $\lambda(R)$. In the following example we specify the failure rate as $\lambda(R) = \frac{1}{a + b(R_0 + R)}$, with $a, b, R_0 > 0$.[4] In this case the optimal maintenance investment level is given by

$$R^*(x) = \max \left[\frac{\sqrt{bux} - u}{b(r - \mu)} - \frac{a + bR_0}{b}, 0 \right]. \tag{12}$$

[4]The results, however, can be easily generalized for other functional forms, such that $\lambda'(R) < 0$ and $\lambda''(R) > 0$.

Note that in some scenarios it is not optimal to make a preventive investment at all. As evident from (12), this may happen when the revenue at the time of investment is small (x is low), the failure rate without preventive investment is already low (a is large), the incidents do not have a large impact on the revenue (u is small), or the preventive investment does not have a large impact on the arrival of incidents (b is small).

The next proposition presents the results for the optimal investment threshold given that the firm spends $R^*(x)$ on mitigation of environmental hazard at the moment of investment.

Proposition 2 *It is only optimal for the firm to enter the market once the project profitability reaches the optimal investment threshold, x^*,which is given by*

$$x^* = \begin{cases} x_R^* & \text{if} \quad \frac{(a+bR_0)^2\left(r-\mu+\frac{u}{a+bR_0}\right)(\beta-1)}{bu\beta} - I_0 - R_0 \leq 0, \\ \frac{\beta}{(\beta-1)}(r-\mu+\frac{u}{a+bR_0})(I_0+R_0) & \text{if} \quad \frac{(a+bR_0)^2\left(r-\mu+\frac{u}{a+bR_0}\right)(\beta-1)}{bu\beta} - I_0 - R_0 > 0, \end{cases} \tag{13}$$

where

$$x_R^* = \frac{1}{b}\left(\frac{(2\beta-1)\sqrt{u}+\sqrt{4(\beta-1)\beta(r-\mu)\left(bI_0-a\right)+u}}{2(\beta-1)}\right)^2. \tag{14}$$

Upon investment the firm chooses the following size of the preventive investment:

$$R^*(x^*) = \begin{cases} R^*(x_R^*) & \text{if} \quad \frac{(a+bR_0)^2\left(r-\mu+\frac{u}{a+bR_0}\right)(\beta-1)}{bu\beta} - I_0 - R_0 < 0, \\ 0 & \text{if} \quad \frac{(a+bR_0)^2\left(r-\mu+\frac{u}{a+bR_0}\right)(\beta-1)}{bu\beta} - I_0 - R_0 \geq 0. \end{cases} \tag{15}$$

Moreover

$$x_R^* > \frac{\beta}{(\beta-1)}(r-\mu+\frac{u}{a+bR_0})(I_0+R_0), \tag{16}$$

meaning that a positive preventive investment results in latter investment.

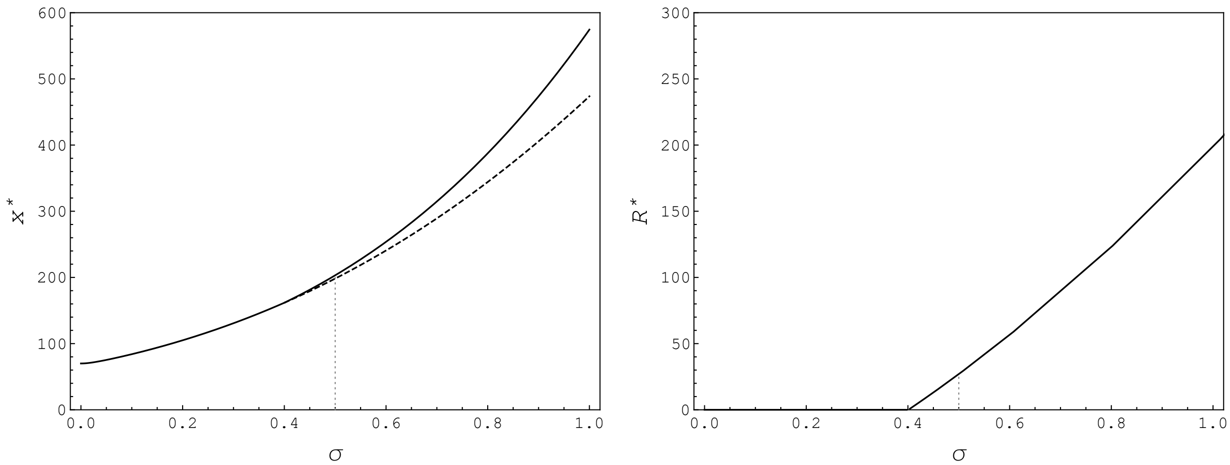
As can be seen from Proposition 2, we can distinguish between two situations (depending on parameter values) that affect the initial investment timing: when the firm invests a positive amount in environmental hazard mitigation ($R^* > 0$) and when it never makes a preventive investment ($R^* = 0$). The crucial difference between these situations is that a positive preventive investment requires an additional cost and, hence, results in a delay of the project investment.

3 Comparative statics

Our baseline parameter values are based on the example from the oil industry. The investment cost and the required preventive investment amount are project specific and vary significantly. We choose to set $I_0 = 600$ and $R_0 = 100$ and later investigate the sensitivity of the model results with respect to these parameter values. The parameter b is a scale parameter that determines the sensitivity of the incidents arrival rate with respect to preventive investment amount and is set to $b = 0.005$ in our baseline case. The frequency of environmental incidents according to the data set of Lundgren and Olsson (2010) is 1 per year for the oil industry. Hence, we set our baseline arrival rate to $\lambda_0 = 1$, which together with $b = 0.005$ and $R_0 = 100$, implies that $a = 0.5$. In industries such as oil and gas environmental incidents appear more frequently than, for example, in utilities. We will address these differences by performing a sensitivity analysis with respect to a and b. As a baseline, we use a conservative average decline in the project value price due to an environmental incident of 1%, implying that $u = 0.01$.

In line with Kellogg (2014), we set the discount rate for the oil project, r, to 9%. In accordance with Costa Lima and Suslick (2006), we set the volatility of the oil project to $\sigma = 0.5$, and the drift to $\mu = 0.01$. Here we use their insight that the volatility of oil projects is larger than that of the oil price process. For example, the oil price volatility 20% translates into the project volatility of $50 - 60\%$, where we choose a more conservative estimate.

To summarize, the baseline parameter values that we use are: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$ $a = 0.5$, $b = 0.005$, $u = 0.01$, $I = 700$, where $I_0 = 600$ and $R_0 = 100$. The figures below illustrate the optimal investment thresholds as well as the optimal preventive investment amount as functions of different parameters. In all figures, the black curve represents the optimal investment threshold for a firm having the option to invest in additional preventive measures, whereas the dashed curve represents the optimal investment threshold in the situation where the firm does not have this option. The vertical dotted lines correspond to the optimal investment threshold and the optimal preventive investment amount for our baseline parameter values.



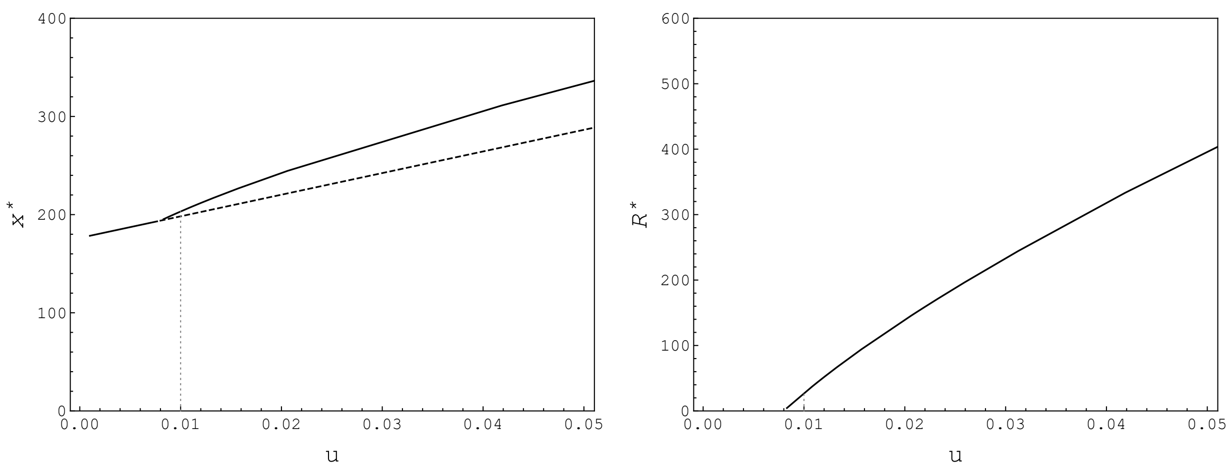
(a) Optimal investment threshold.

(b) Optimal preventive investment size.

Figure 1: Optimal investment threshold and optimal preventive investment size as functions of σ. [Parameter values: $r = 0.09$, $\mu = 0.01$ $a = 0.5$, $b = 0.005$, $u = 0.01$, $I_0 = 600$ and $R_0 = 100$.]

Figure 1a is according to the standard result in the real options literature that a larger volatility leads to a larger investment threshold, which means that the firm requires a larger level of profitability to undertake the investment. As a result, the firms is also able to allocate a larger investment amount to preventive measures, as illustrated in Figure 1b.

Figure 2 shows how the investment decision is affected by a change in the average jump size u.



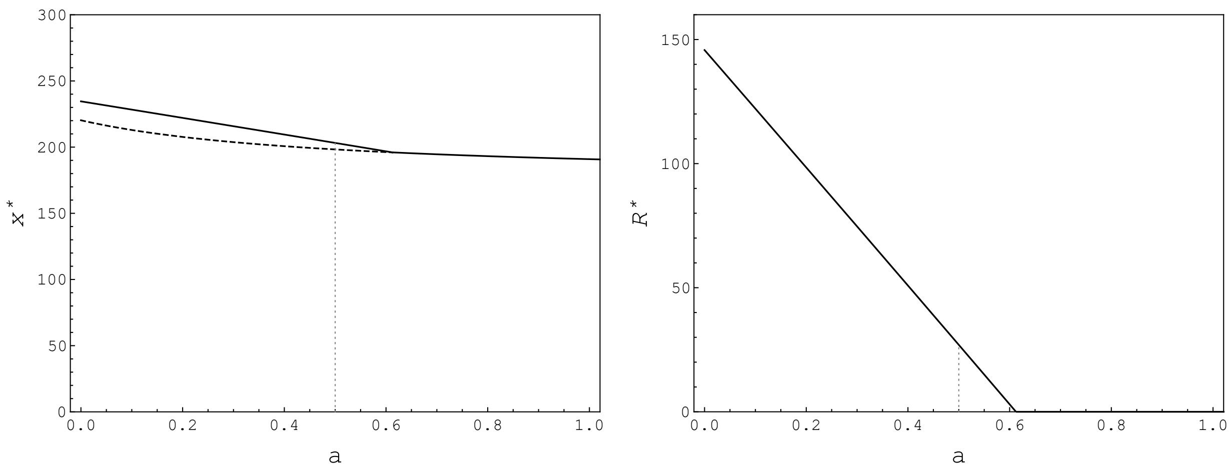
(a) Optimal investment threshold.

(b) Optimal preventive investment size.

Figure 2: Optimal investment threshold and optimal preventive investment size as functions of u. [Parameter values: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$ $a = 0.5$, $b = 0.005$, $I_0 = 600$ and $R_0 = 100$.]

As can be seen, increased average damage makes the project less attractive and therefore delays project investment. In addition, increased average damage also raises the need for a larger preventive investment, which in turn also delays project investment.

Figures 3 and 4 depict the effect of the parameters that affect the arrival rate of the environmental incidents. Here, a is inversely related to the arrival rate of incidents also when the firm does not undertake any voluntary investment, whereas b reflects the sensitivity of the inverse arrival rate to the preventive investment amount.

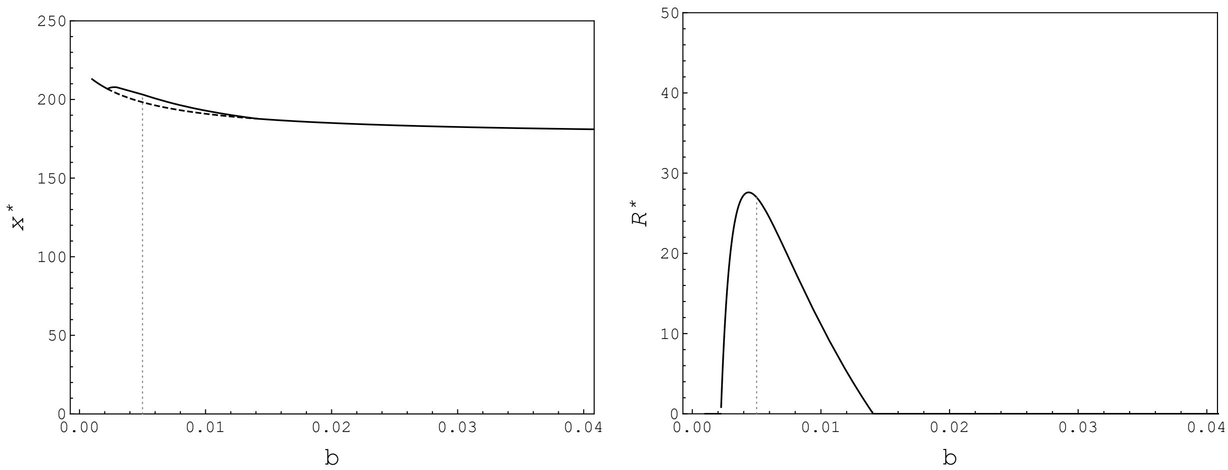


(a) Optimal investment threshold.

(b) Optimal preventive investment size.

Figure 3: Optimal investment threshold and optimal preventive investment size as functions of a. [Parameter values: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$, $b = 0.005$, $u = 0.01$, $I_0 = 600$ and $R_0 = 100$.]

If a is large, implying that the arrival rate of incidents is small even without an additional investment, the preventive investment becomes less necessary. This leads to a smaller investment cost and, thus, earlier investment.

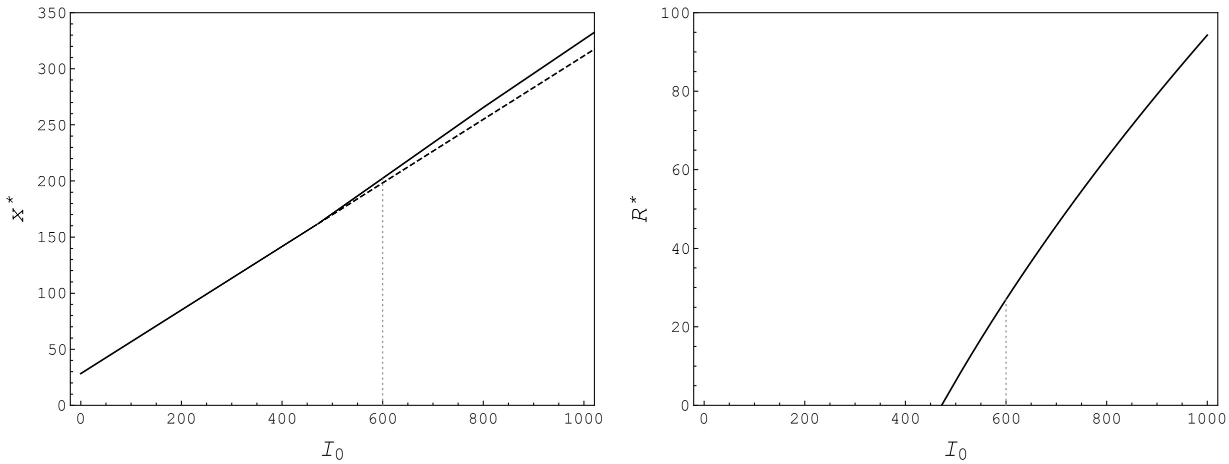


(a) Optimal investment threshold.

(b) Optimal preventive investment size.

Figure 4: Optimal investment threshold and optimal preventive investment size as functions of b. [Parameter values: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$ $a = 0.5$, $u = 0.01$, $I_0 = 600$ and $R_0 = 100$.]

An increase in b results in a non-monotonic effect on both the investment threshold and the optimal size of the preventive investment. When b is small, an increase in preventive investment leads to a smaller effect on the arrival rate of incidents. Then a larger b raises the effectiveness of preventive investment and, therefore, the firm invests more. For large values of b, the arrival rate is more sensitive to changes in preventive investment size, resulting in the additional effect that already a small preventive investment substantially reduces the incidents' arrival rate. Therefore, in this b-domain, the optimal size of preventive investment is decreasing in b. The optimal investment threshold follows the behavior of the optimal R. A larger preventive investment makes the project more expensive, which induces a later investment and vice versa.



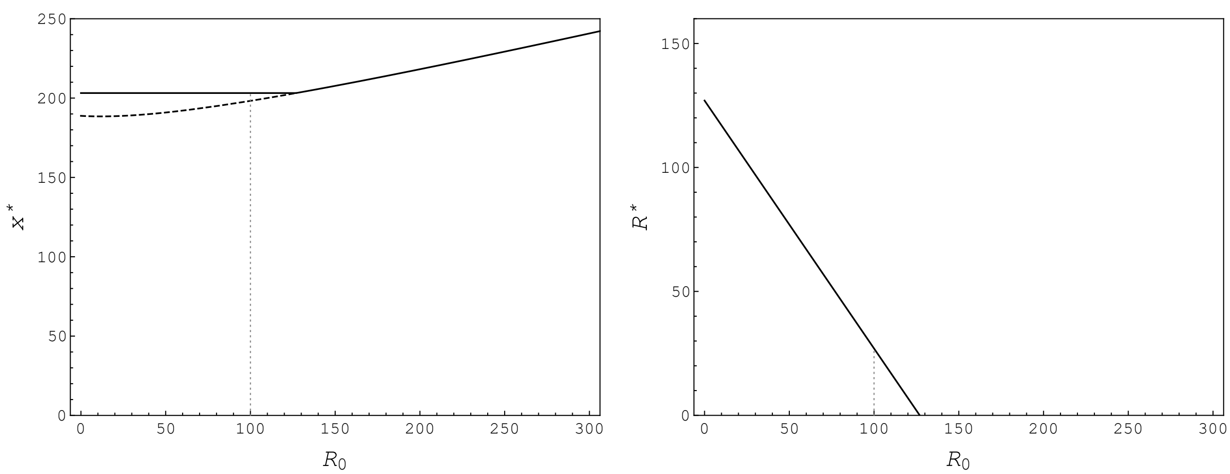
(a) Optimal investment threshold.

(b) Optimal preventive investment size.

Figure 5: Optimal investment threshold and optimal preventive investment size as functions of I_0. [Parameter values: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$ $a = 0.5$, $b = 0.005$, $u = 0.01$, and $R_0 = 100$.]

Figure 5 shows the effect of the project investment amount. A larger investment amount makes the firm investing later, implying that at the moment of investing the profitability of the project is larger. This enlarges the desire to avoid environmental incidences, resulting in additional voluntary preventive investments for I_0 large enough.

Figure 6 illustrates the effect of a change in the required preventive investment amount on firm's investment timing and size decisions.



(a) Optimal investment threshold.

(b) Optimal preventive investment size.

Figure 6: Optimal investment threshold and optimal preventive investment size as functions of R_0. [Parameter values: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$ $a = 0.5$, $b = 0.005$, $u = 0.01$, and $I_0 = 600$.]

In fact, total preventive investment being equal to $R_0 + R^*$ is constant. If the required preventive investment amount is large, less additional preventive investment is needed to reduce the arrival rate of incidents. Therefore, a larger R_0 negatively affects R^*. If total preventive investment goes up, project value increases and the firm invests later. This explains why x^* increases in R_0.

4 Discretion over timing of preventive investment

In the previous section the firm is obliged to undertake preventive investment at the time it initiates the project. Thus, both investment decisions occur simultaneously.

In this section, the firm have the opportunity to undertake the voluntary preventive investment after the project investment already has been undertaken. Consequently, we have to determine:

1. When to invest in the project;
2. Upon investment, when to make the voluntary preventive investment.

After the firm has invested in the project, it is subject to accidents. Since these reduce the project's profitability, the number of accidents occurred so far is a relevant input to the voluntary preventive investment decision. Consequently, in order to formulate an optimal preventive policy, the firm needs to keep record of the number of accidents occurred so far.

Next to the timing of the preventive investment, we still also have to determine the investment size. The relevant value function to determine the timing and the size of the voluntary preventive investment is the one that arises after the firm has undertaken the project investment. This value function can be expressed as follows:

$$\begin{aligned} V_1(x,n) &= \sup_{\tau, R \geq 0} \mathbb{E}_{x,n} \left[\int_0^{\tau} \mathrm{e}^{-rt} X_t \prod_{i=1}^{N_t} (1-U_i) dt - R\mathrm{e}^{-r\tau} \right. \\ &+ \left. \int_{\tau}^{\infty} \left(X_t \prod_{i=1}^{N_\tau} (1-U_i) \prod_{i=1}^{N^R_{t-\tau}} (1-U_i) \right) \mathrm{e}^{-rt} dt \right], \end{aligned} \tag{17}$$

where N^R is the Poisson process with intensity rate $\lambda(R)$. To find the optimal timing of the project investment, we have to maximize the project value, which includes the option to invest in voluntary preventive investment. The corresponding optimization problem can be expressed as

$$V_2(x) = \sup_{\tau_1 \leq \tau_2, R \geq 0} \mathbb{E}_x \left[\int_{\tau_1}^{\tau_2} \mathrm{e}^{-rt} \left(X_t \prod_{i=1}^{N_{t-\tau_1}} (1 - U_i) \right) dt - (I_0 - R_0)\mathrm{e}^{-r\tau_1} \right. \tag{18}$$

$$+ \left. \int_{\tau_2}^{\infty} \left(X_t \prod_{i=1}^{N_{\tau_2-\tau_1}} (1 - U_i) \prod_{i=1}^{N^R_{t-\tau_2}} (1 - U_i) \right) \mathrm{e}^{-rt} dt - Re^{-r\tau_2} \right], \tag{19}$$

in which τ_1 denotes the time of the project investment, and τ_2 is the time of the preventive investment. Of course it has to hold that $\tau_1 \leq \tau_2$, as the preventive investment only makes sense after the project has started.

Similar arguments as the ones used in the previous section lead us to conclude that solving (17) is the same as solving:

$$V_1(x, n) = \frac{x}{r + u\lambda(0) - \mu} + \sup_{\tau, R \geq 0} \mathbb{E}_{x,n} \left[e^{-r\tau} h(X_\tau, N_\tau; R) \right],$$

where

$$h(x, n; R) = (1 - u)^n x \left(\frac{1}{r + u\lambda(R) - \mu} - \frac{1}{r + u\lambda(0) - \mu} \right) - R. \tag{20}$$

Using the strong Markov property, it follows that the problem described by V_2 is related with V_1 as follows:

$$V_2(x) = \sup_{\tau} E_x[e^{-r\tau} (V_1(X_\tau, 0) - I_0 - R_0)]. \tag{21}$$

For the sake of clarity, we note that in (17) we consider that $N_0 = n$, whereas in (21) $N_0 = 0$. Before the project investment there are no accidents, and thus this means that at the project investment time, the number of accidents is zero.

The standard real options analysis learns that the solution of the optimization problem (21) is as follows:

$$V_2(x) = \begin{cases} Cx^\beta & x < x_I, \\ V_1(x, 0) - I_0 - R_0 & x \geq x_I. \end{cases} \tag{22}$$

where x_I denotes the optimal project investment threshold. As usual, C and x_I can be derived using the smooth fit conditions:

$$Cx_I^\beta = V_1(x_I, 0) - I_0 - R_0, \quad \beta C x_I^{\beta-1} = \frac{dV_1(x, 0)}{dx}|_{x=x_I}.$$

Therefore in order to solve the problem for V_2, we need first to solve for V_1. And this means that we need to maximize with respect to the investment size R and the preventive investment time τ_2.

We start by regarding the maximization with respect to R: it follows from (20), that the effect of R on the terminal value has an ambiguous effect, as in the baseline model. The firm will invest an amount R^* given by

$$R^*(x,n) = \underset{R\geq 0}{\operatorname{argmax}}\, h(x,n;R). \tag{23}$$

For the particular case $\lambda(R) = \frac{1}{a+b(R+R_0)}$, the firm will invest the following quantity in the preventive center:

$$R^*(x,n) = \frac{-u+\sqrt{b(1-u)^n ux}}{b(r-\mu)} - \frac{a+bR_0}{b},$$

if $\frac{-u+\sqrt{b(1-u)^n ux}}{b(r-\mu)} - \frac{a+bR_0}{b} > 0$. If this condition does not hold, the preventive investment won't take place.

This leads straightforwardly to the following result:

Proposition 3 *The preventive investment amount R^* decreases with the number of incidents and increases in the revenue.*

Regarding the optimal time to undertake the voluntary preventive investment, we note that the value of the corresponding investment option, V_1, can be obtained by solving the following equation:

$$rv(x,n) - \mathcal{L}v(x,n) = 0, \tag{24}$$

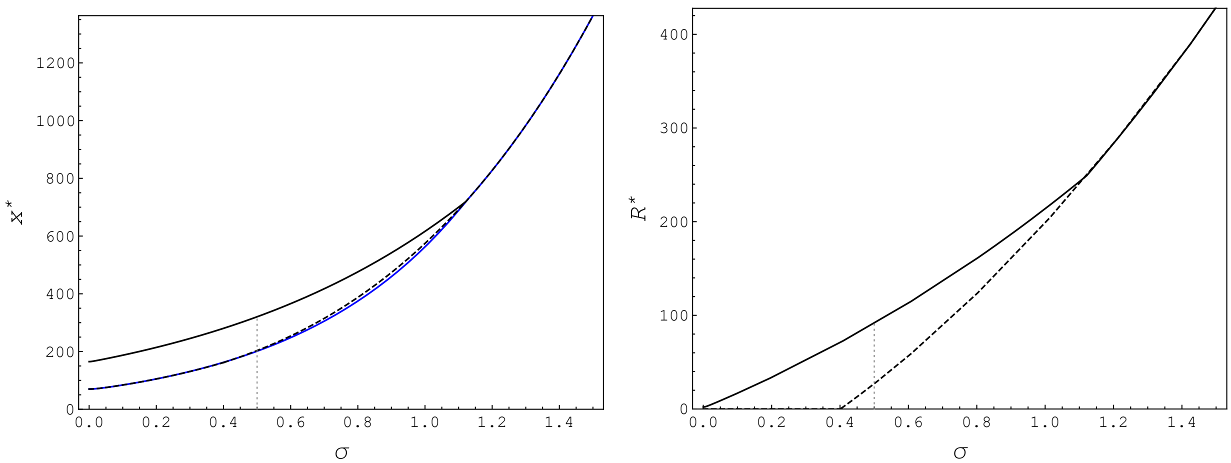
with $\mathcal{L}$ given by

$$\mathcal{L}v(x,n) = \mu x v'(x,n) + \frac{\sigma^2}{2}x^2 v''(x,n) + \lambda_0(v(x,n+1) - v(x,n)),$$

where the last term represents the expected revenue loss in case one accident occurs. An analytical solution to differential equation (24) is not possible to find, as discussed in Nunes *et al.* (2020). Therefore, in the Appendix we present a quasi-analytical solution for the problem described in (17).

4.1 Comparative statics

In this section, we present the comparative statics result for the model where preventive investment and initial investment do not necessarily have to occur at the same time. In the subsequent figures the blue solid threshold curve represents the *initial investment threshold* and the black solid curve corresponds to the *preventive investment decision (timing and size)*. The black dashed curve is the investment decision when both investments occur at the same time (baseline model). Again, the vertical dotted lines correspond to the optimal investment threshold and the optimal preventive investment amount for our baseline parameter values.

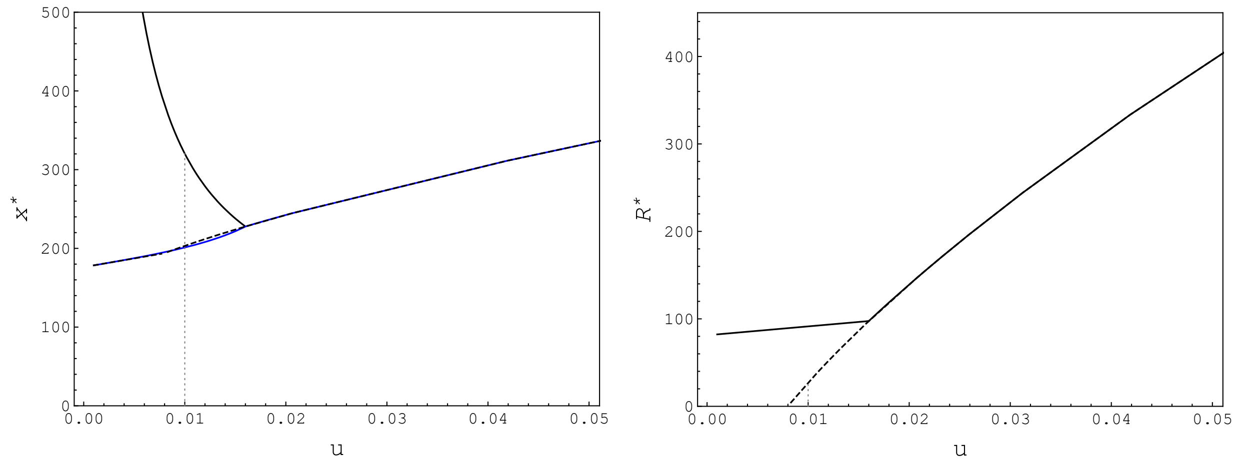


(a) Optimal investment thresholds.

(b) Optimal preventive investment size.

Figure 7: Optimal investment thresholds and optimal preventive investment size as functions of σ. [Parameter values: $r = 0.09$, $\mu = 0.01$ $a = 0.5$, $b = 0.005$, $u = 0.01$,$I_0 = 600$ and $R_0 = 100$.]

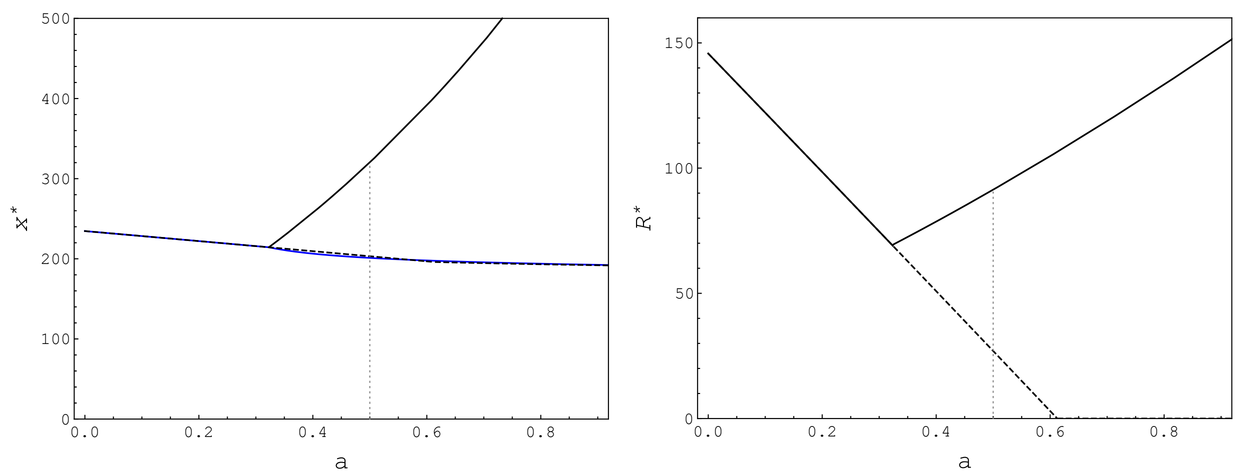
Figure 7 shows that as uncertainty increases, both investments occur later. However, the gap between preventive investment and initial investment becomes smaller. This is because for higher uncertainty the initial investment occurs when the project profitability is larger. This implies that environmental incidences will cause a larger reduction of the firm value, which increases the need for preventive investment. Also, the initial investment occurs earlier in the extended model in comparison to the baseline model in case the firm indeed decides to undertake the preventive investment later. This is because the possibility to undertake the preventive investment later increases the initial investment profitability. Therefore, the firm is more eager to invest and does it earlier. Another reason for earlier initial investment occurs when in the baseline model the firm decides to undertake voluntary preventive investment. This increases the investment outlay in the baseline model and therefore the firm invests later in that case. It is also important to note that preventive investments increase in size when they are undertaken at a later point in time than the project investment. The obvious explanation is that project profitability is larger at the moment the firm makes the preventive investment.



(a) Optimal investment thresholds. (b) Optimal preventive investment size.

Figure 8: Optimal investment thresholds and optimal preventive investment size as functions of u. [Parameter values: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$ $a = 0.5$, $b = 0.005$, $I_0 = 600$ and $R_0 = 100$.]

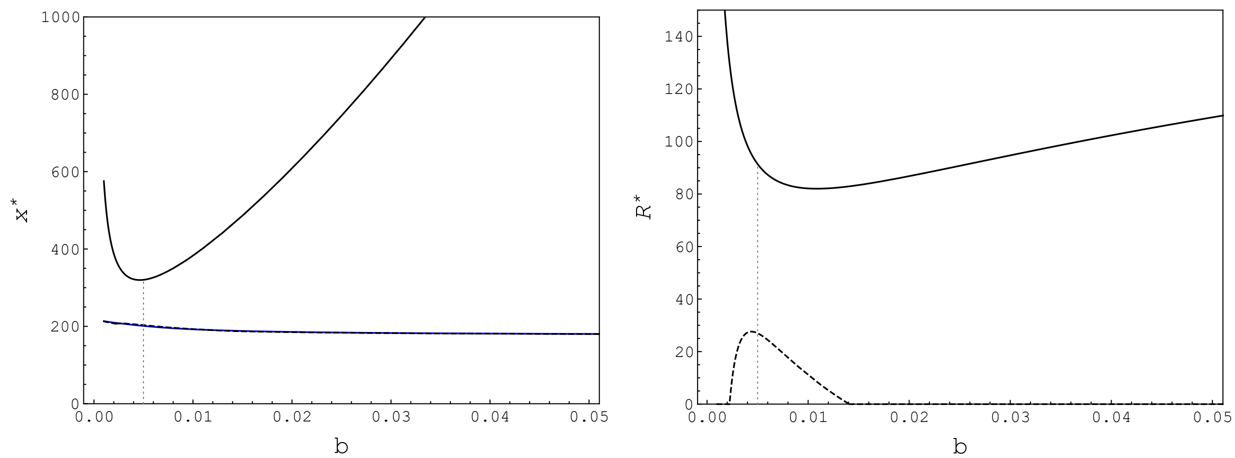
As the impact of the incidents, u, increases, preventive investment timing decreases, whereas its size increases, as depicted in Figure 8. The more damaging the incidents are, the more eager is the firm to invest more in preventing them. Consequently, the gap between initial investment and preventive investment decreases in u.



(a) Optimal investment thresholds. (b) Optimal preventive investment size.

Figure 9: Optimal investment thresholds and optimal preventive investment size as functions of a. [Parameter values: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$, $b = 0.005$, $u = 0.01$, $I_0 = 600$ and $R_0 = 100$.]

As seen in Figure 9, the larger is a, the lower is the likelihood of incidents' arrivals. Then a preventive investment becomes less necessary, so it happens later. The latter means that the project profitability is higher at the moment of the preventive investment, and therefore the firm will increase the preventive investment size. As a result, the gap in timing between preventive investment and initial investment increases in a.

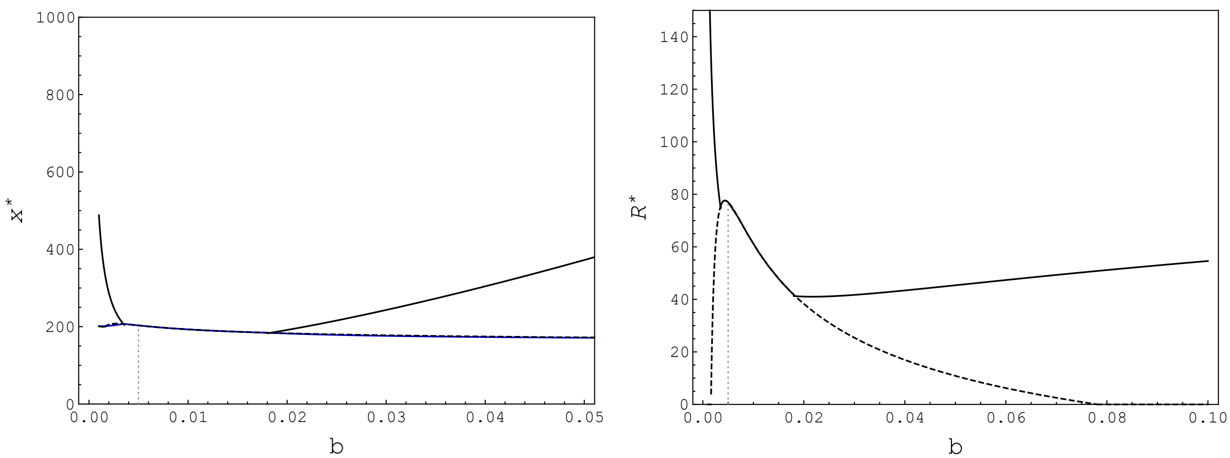


(a) Optimal investment thresholds.

(b) Optimal preventive investment size.

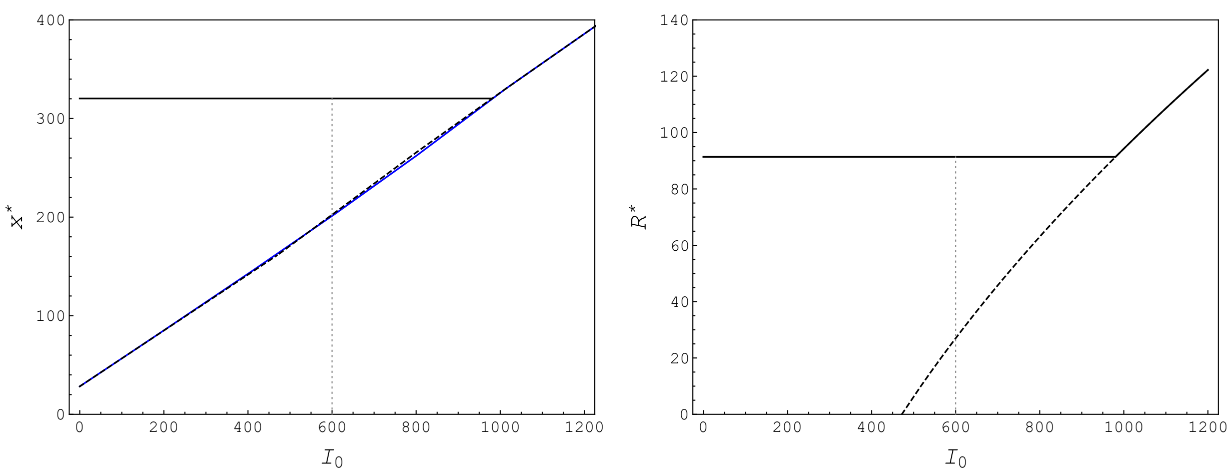
Figure 10: Optimal investment thresholds and optimal preventive investment size as functions of b. [Parameter values: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$ $a = 0.5$, $u = 0.01$, $I_0 = 600$ and $R_0 = 100$.]

The preventive investment threshold is non-monotonic in b, as shown in Figure 10. On the one hand, the larger is b the more efficient the preventive investment is, which incentivizes the firm to invest sooner. On the other hand, the larger is b the more efficient is the mandatory investment, R_0. In this case, the firm does not need to undertake additional voluntary preventive investments as the required investment reduces the incident arrivals by a lot already. Therefore, it invests later, and thus more since there are more revenues to protect at that time. In this example the mandatory investment $R_0 = 100$ is so large that it always makes sense to wait with preventive investment and not to do it immediately after the initial investment. This is obviously not the case if we decrease R_0, as confirmed in Figure 11 where $R_0 = 50$. Here it holds that for intermediate levels of b both investments happen at the same time. This indicates the policies regarding mandatory measures to prevent incidents have a substantial impact on the investment strategies of firms.



(a) Optimal investment thresholds. (b) Optimal preventive investment size.

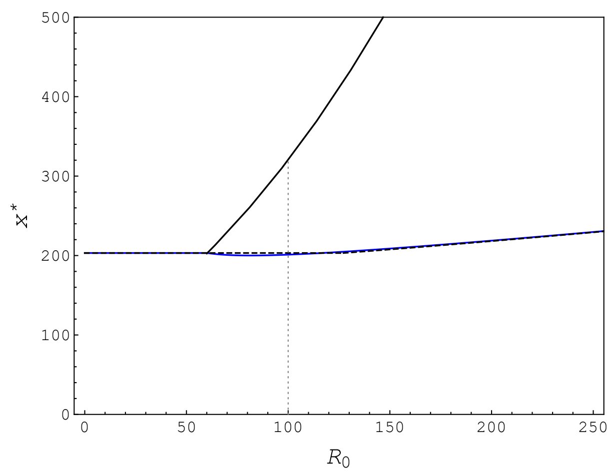
Figure 11: Optimal investment thresholds and optimal preventive investment size as functions of b. [Parameter values: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$ $a = 0.5$, $u = 0.01$, $I_0 = 600$ and $R_0 = 50$.]



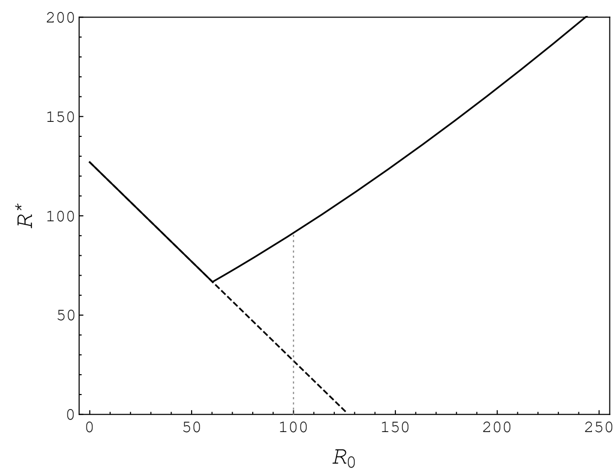
(a) Optimal investment thresholds. (b) Optimal preventive investment size.

Figure 12: Optimal investment thresholds and optimal preventive investment size as functions of I_0. [Parameter values: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$ $a = 0.5$, $b = 0.005$, $u = 0.01$, and $R_0 = 100$.]

Figure 12 depicts the effect of different levels of I_0. The larger is I_0 the later the initial investment happens, which is according to intuition. Furthermore, I_0 does not affect the preventive investment, in case the investment times are separated. However, this is not the case when they occur at the same time. Then we have the baseline model result that an increase in I_0 induces a larger preventive investment.



(a) Optimal investment thresholds.



(b) Optimal preventive investment size.

Figure 13: Optimal investment thresholds and optimal preventive investment size as functions of R_0. [Parameter values: $r = 0.09$, $\sigma = 0.5$, $\mu = 0.01$ $a = 0.5$, $b = 0.005$, $u = 0.01$, and $I_0 = 600$.]

Figure 13 shows that the larger is R_0 the less incentives has the firm to invest additionally in prevention. Thus, preventive investment happens later (and, thus, is of larger size).

5 Conclusion

Ample industrial projects exist that go along with risks of having environmental incidents. Although firms need not always be liable, the common denominator is that such incidents will reduce firm value. It is therefore important to analyze the option to invest in a project together with the option to undertake preventive investments that reduce the probability of occurrence of such environmental incidents.

In this paper, we first consider project investment in the case that preventive investment should occur at the same time. We find that preventive investments are large when (i) the project revenue is large, (ii) the environmental incidents potentially cause a huge reduction of firm value, and (iii) when preventive investment substantially decreases the probability of environmental damage occurrence. In these situations the firm is inclined to invest more when it wants to start such an industrial project, and this will lead to a considerable delay.

Next, we take a more sequential view on the investment strategies in the sense that we give the firm the option to undertake the preventive investment at a later point in time than the project investment. We identify two new effects if we compare the investment outcome to our first model. First, the firm decides to start the industrial project sooner, because the initial sunk costs are smaller.

Second, the firm undertakes a larger preventive investment. This is because preventive investment occurs later, meaning that then the project's revenue is larger. Hence, there is more to lose for this firm, which thus incentivizes the desire to increase the preventive investment.

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Appendix

A Numerical approximation

We propose a quasi-analytical solution for the problems described in (17) and, consequently, (21), based on a *truncation method*. By truncated problem we mean that in the definition of V_1 we impose an additional constraint on the admissible stopping times. In this case, we assume there is a number of accidents sufficiently large for which the corresponding value function is zero, regardless of the value x of the project. We let $\overline{N}$ denote such value and we let $\overline{V}_1^*(x,n)$ denote the value function when the project value is x, and n accidents have already occurred, for the truncated problem. Furthermore, we let $\tau_{\overline{N}}$ be the time at which the $\overline{N}$ accident occurs.

The truncated problem corresponding to (17) is defined as follows:

$$\overline{V}_1^*(x,n) = \sup_{\tau \leq \tau_{\overline{N}}, R \geq 0} \mathbb{E}_{x,n} \left[\int_0^{\tau} \mathrm{e}^{-rt} X_t \prod_{i=1}^{N_t} (1-U_i) dt - R\mathrm{e}^{-r\tau} \right. \\ \left. + \int_{\tau}^{\infty} \left(X_t \prod_{i=1}^{N_\tau} (1-U_i) \prod_{i=1}^{N_{t-\tau}^{R^*}} (1-U_i) \right) \mathrm{e}^{-rt} dt \right], \tag{25}$$

which means that in (25) the optimal stopping time is bounded above by $\tau_{\overline{N}}$, whereas in (17) it does not have such restriction. The meaning of such truncation comes from the following. Let $\tilde{h}(x,n) = h(x,n;R^*(x,n))$, so that $\tilde{h}(x,n)$ represents the terminal cost if the firm decides to invest in the preventive center when the profit is x, n accidents have already occurred and the firm invests an optimal value $R^*(x,n)$. Note that for each x, there is a number of accidents $n(x)$ for which $\tilde{h}(x,n) < 0, \forall n \geq n(x)$. Therefore, intuitively one expects to have $\overline{N} = \sup\{n(x) : \tilde{h}(x,n) < 0, \forall x, \forall n \geq n(x)\}$.

We use the notation $\bar{}$ to denote that we are using the truncation method, and $*$ to denote that we have optimized with respect to the preventive investment. We assume that $\overline{V}_1^*(x,\overline{N}) = 0, \forall x$. Moreover, we let $\overline{x}_n^*$ denote the level of the project that triggers the preventive investment, given that n accidents have already occurred and $n \leq \overline{N}$.

As the h function, as defined on (20), is decreasing in n, for a fixed x, it follows that $\overline{x}_n^*$ increases in n, i.e., for $n_1 < n_2$, the firm requires a larger profit to make the preventive investment when the number of accidents occurred so far is n_2 than when it is n_1. Therefore, one may expect waiting/investment regions as depicted in Figure 14.

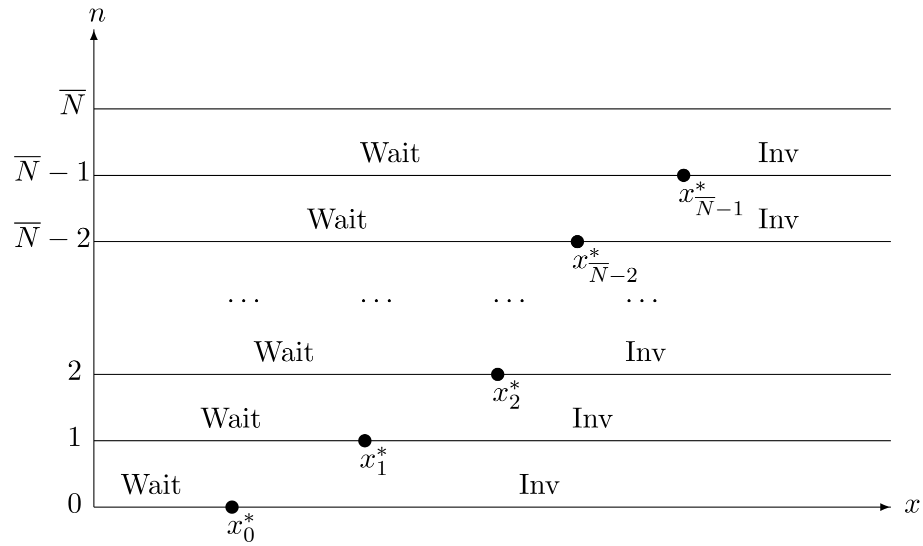

Figure 14: Optimal strategy for preventive investment.

Following Nunes *et al.* (2020), after some manipulations in order to consider our set-up, we end up with the following expressions for the solution of problem (25).

$$\overline{V}_1^*(x, \overline{N}-i) = \frac{x}{r+u\lambda(0)-\mu} + \begin{cases} \sum_{j=0}^{i-1} A_{N-i,j}(\ln x)^j x^{d_2} & x < \overline{x}^*_{\overline{N}-i}, \\ h'\left(x, \overline{N}-i; R^*(x, \overline{N}-i)\right) & x \geq \overline{x}^*_{\overline{N}-i}, \end{cases} \tag{26}$$

with

$$A_{m,j} = -\frac{2\lambda_0}{\sigma^2} \sum_{l=j-1}^{\overline{N}-2-m} (-1)^{l-j+1} \frac{l!}{j!} \frac{A_{m+1,l}}{(d_2-d_1)^{l+2-j}}, \tag{27}$$

for $j \geq 1$,and

$$A_{n,0} = (x_n^*)^{-d_2}\tilde{h}(x,n) - \left(A_{n,1} ln(x_n^*) + \left(\sum_{j=2}^{\overline{N}-n-1} j A_{n,j} \left(ln(x) \right)^j \right) \chi_{\{n \neq \overline{N}-2\}} \right) \chi_{\{n \neq \overline{N}-1\}},$$

and the thresholds satisfy the equations

$$\begin{aligned} &(x_n^*)^{d_2} \left(A_{n,1} + \left(\sum_{j=2}^{\overline{N}-n-1} j A_{n,j} \left(ln(x_n^*) \right)^{j-1} \right) \chi_{\{n \neq \overline{N}-2\}} \right) \chi_{\{n \neq \overline{N}-1\}} \\ &+(1-u)^{\overline{N}-1}(d_2-1)x_n^* \left(\frac{1}{r+\lambda(R^*(x_n^*,n))u-\mu} - \frac{1}{r+\lambda(0)u-\mu} \right) - d_2 R^*(x_n^*,n) = 0. \end{aligned} \tag{28}$$

In the above equations, d_1 and d_2 are the roots of the characteristic polynomial $\frac{\sigma^2}{2}d(d-1)+\mu d-(r+\lambda)$:

$$d_1 = \frac{\left(\frac{\sigma^2}{2}-\mu\right) + \sqrt{\left(\frac{\sigma^2}{2}-\mu\right)^2 + 2\sigma^2(r+\lambda)}}{\sigma^2} > 1 \text{ and } d_2 = \frac{\left(\frac{\sigma^2}{2}-\mu\right) - \sqrt{\left(\frac{\sigma^2}{2}-\mu\right)^2 + 2\sigma^2(r+\lambda)}}{\sigma^2} < 0.$$

The analytical computation of the coefficients $A_{m,j}$ is not possible, but computationally does not lead to any problem. Consequently, one can only find the thresholds $\overline{x}^*_{\overline{N}-i}$ (for $i = 2, \ldots, \overline{N}$) and $\overline{x}^*_I$ using numerical values.

B Proofs

Proof of Proposition 1 The value matching and smooth pasting conditions lead to the following system of equations:

$$\begin{cases} A{x^*}^{\beta} = v(x^*, R^*(x^*)), \\ \beta A{x^*}^{\beta-1} = \frac{\partial}{\partial x^*}v(x^*, R^*(x^*)) + \frac{\partial}{\partial R^*(x^*)}v(x^*, R^*(x^*))\frac{\partial R^*(x^*)}{\partial x^*}, \end{cases} \tag{29}$$

where in the last equation we use the total derivative rule in order to compute $\frac{\partial v(x^*, R^*(x))}{\partial x^*}$.

Taking into account the definition of R^*, it follows that $\frac{\partial}{\partial R^*(x^*)}v(x^*, R^*(x^*)) = 0$, and therefore the system (29) becomes

$$\begin{cases} A(x^*)^{\beta} = \dfrac{x^*}{r - \mu + \lambda(R^*(x^*))u} - I_0 - R_0 - R^*(x^*), \\ \beta A(x^*)^{\beta-1} = \dfrac{1}{r - \mu + \lambda(R^*(x^*))u}. \end{cases} \tag{30}$$

Simplifying yields the expression for A

$$A = \frac{1}{\beta(x^*)^{\beta-1}(r - \mu + \lambda(R^*(x^*))u)}, \tag{31}$$

and the following implicit equation for the optimal threshold

$$\frac{x^*}{r - \mu + \lambda(R^*(x^*))u}\left(1 - \frac{1}{\beta}\right) - I_0 - R_0 - R^*(x^*) = 0. \tag{32}$$

Note that (32) can be represented as the difference between the value of investing and the value of waiting $\dfrac{x^*}{r - \mu + \lambda(R^*(x^*))u} - I_0 - R_0 - R^*(x^*) - A(x^*)^{\beta}$, which at the optimal investment threshold should be equal to 0 . Thus, for $x < x^*$ it should hold that waiting is more valuable and, thus, $\dfrac{x^*}{r - \mu + \lambda(R^*(x^*))u} - I_0 - R_0 - R^*(x^*) - A(x^*)^{\beta} < 0$. Thus, this allows us to identify the following condition for the optimal threshold

$$\frac{(x^* - h)}{r - \mu + \lambda(R^*(x^* - h))u}\left(1 - \frac{1}{\beta}\right) - I_0 - R_0 - R^*(x^* - h) < 0 \text{ for } h \in (0, x^*). \tag{33}$$

Proof of Proposition 2 From (12), it follows that $R^*(x)$ can be written as

$$R^*(x) = \begin{cases} 0, & \text{if} \quad x < \hat{x}, \\ \frac{\sqrt{bux} - u}{b(r-\mu)} - \frac{a + bR_0}{b} & \text{if} \quad x \geq \hat{x}, \end{cases} \tag{34}$$

where

$$\hat{x} = \frac{((a + bR_0)(r - \mu) + u)^2}{bu}. \tag{35}$$

Plugging in the expressions for $R^*(x)$ and $\lambda(R)$ in (32), we get that x^* is the solution of $F(x) = 0$, where

$$F(x) = \begin{cases} \left(\frac{\beta-1}{\beta}\right) \frac{x}{r-\mu+\frac{u}{a+bR_0}} - I_0 - R_0 & \text{if} \quad x < \hat{x}, \\ \frac{a}{b} + \left(\frac{\beta-1}{\beta}\right) \frac{x(-u+\sqrt{bux})}{1-u+\sqrt{bux})(r-\mu)} + \frac{u-\sqrt{bux}}{b(r-\mu)} - I_0 & \text{if} \quad x \geq \hat{x}. \end{cases} \tag{36}$$

As $I_0 > 0$, we can conclude that (36) has at least one solution, because $F(x)\Big|_{x\geq\hat{x}}$ is U-shaped parabola. Therefore

$$x^* = \min\{x > 0 : F(x) = 0\}, \tag{37}$$

so that x^* is always the smallest positive root of $F(x) = 0$.

Since $F(\hat{x}) = \left(\frac{\beta-1}{\beta}\right) \frac{((u+(r-\mu)(a+bR_0))^2}{bu(r-\mu+\frac{1}{a+bR_0})}$, it is optimal to invest $R^* > 0$, when $\frac{(a+bR_0)^2\left(r-\mu+\frac{u}{a+bR_0}\right)(\beta-1)}{bu\beta} - I_0 - R_0 \leq 0$, and zero otherwise. Given this condition, solving for x yields the result presented in (13-15). We note that we are able to prove that $R^*(x)$ is one and only zero of the first order derivative of v w.r.t. R, and that this is exactly the maximizer defined in equation (23).

Finally, (16) holds in view of the definition of F, (36), and the fact that F has at least one zero.