NUMERICAL APPROXIMATION OF CAUCHY PROBLEMS FOR MULTIDIMENSIONAL PDES WITH UNBOUNDED COEFFICIENTS ARISING IN FINANCIAL MATHEMATICS

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ABSTRACT. In this article, we study the numerical approximation of the solution of the Cauchy problem for a multidimensional linear parabolic PDE of second order, with unbounded time and space-dependent coefficients. The PDE free term and the initial data are also allowed to grow. Under the assumption that the PDE does not degenerate, using the L^2 theory of solvability in weighted Sobolev spaces, the PDE problem's weak solution is approximated in space, with the use of finite-difference methods. Making also use of finite differences (with both the explicit and implicit schemes), the approximation in time is considered in abstract spaces for evolution equations, and then specified to the second-order parabolic PDE problem. The rate of convergence is estimated for the approximation in space and time.

1. Introduction

In this article, we study the numerical approximation of second-order linear parabolic PDEs with unbounded coefficients on the strip $[0, T] \times \mathbb{R}^d$, with T a positive constant.

Multidimensional PDE problems arise in Financial Mathematics and in Mathematical Physics. We are mainly motivated by the application to a class of stochastic models in Financial Mathematics, comprising the non path-dependent options, with fixed exercise, written on multiple assets (*basket options*, *exchange options*, *compound options*, European options on future contracts and foreign-exchange, and others), and also, to a particular type of path-dependent options, the Asian options (see, e.g., [13, 27]).

Let us consider the stochastic modelling of a multi-asset financial option of European type, within the framework of a multidimensional version of Black-Scholes model, where the vector of asset appreciation rates and the volatility matrix are taken time and space-dependent. Making use of a Feynman-Kac type formula, pricing this option can be reduced to solving the Cauchy problem (with terminal condition) for a degenerate multidimensional parabolic PDE, with unbounded coefficients and null term (see, e.g., [13]). Therefore, alternatively to approximating the option price with Monte Carlo simulation, we can approximate the solution of the correspondent PDE problem.

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In this article, we tackle the challenge posed to the approximation by the unboundedness of the PDE coefficients, under the strong assumption that the PDE does not degenerate. We study the numerical approximation of the solution of the Cauchy problem[1]

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ in } \mathbb{R}^d, \tag{1.1}$$

where

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c(t,x), \quad i,j = 1,\dots,d$$

is a partial differential operator with real coefficients,[2] for each $t \in [0,T]$, with $T \in (0,\infty)$ a constant, uniformly elliptic with respect to the space variables, $Q = [0,T] \times \mathbb{R}^d$, and f and g are given real-valued functions. We allow the growth in space of the first and second-order coefficients in L (linear and quadratic growth, respectively), and of the data f and g (polynomial growth).

When problem (1.1) is considered in connection with the Black-Scholes modelling of a financial option, we see that the growth of the vector SDE coefficients in the underlying financial model is appropriately matched. Also, by setting the problem with this generality, we cover the general case where the asset appreciation rate vector and the volatility matrix are taken time and space-dependent. Finally, by letting the initial data g non-specified, a large class of pay-off functions can be considered in the underlying financial derivative modelling. The free term f is included to further improve generality.

In order to facilitate the approach, we avoid numerical method sophistication, and make use of basic one-step finite-difference schemes.

The numerical methods and possible approximation results are strongly linked to the theory on the solvability of the PDEs. In this article, we make use of the L^2 theory of solvability of linear PDEs in weighted Sobolev spaces. In particular, we consider the PDE solvability in the deterministic special case of a class of weighted Sobolev spaces introduced by O. G. Purtukhia [19, 20, 21, 22], and further generalized by I. Gyöngy and N. V. Krylov [9], for the treatment of linear SPDEs. By considering discrete versions of these spaces, we set a suitable discretized framework and investigate the PDE approximation.

The finite-difference method for approximating PDE is a well developed area, which has been extensively researched since the first half of the last century. We refer to [26] for a brief summary of the method's history, and also for the references of the seminal work by R. Courant, K. O. Friedrichs and H. Lewy, and further major contributions by many others.

Also in [26], we can find the numerical study, making use of finite differences, of the Cauchy problem for a general multidimensional linear parabolic PDE of order $m \geq 2$, with bounded time and space-dependent coefficients. This study is pursued in the framework of the classical approach.

Although the theory can be considered reasonably complete since three decades ago, some important research continues. We mention, just as an example, the recent works [11, 12].

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[1] Instead of the terminal-value formulation arising from the financial model, we consider the more standard initial-value formulation. Clearly, one problem can be transformed in the other by a simple change of the time variable.

[2] The operator L is written with the usual summation convention.

The finite-difference method was early applied to financial option pricing, the pioneering work being due to M. Brennan and E. S. Schwartz in 1978, and was, since then, widely researched in the context of the financial application, and extensively used by practitioners. For the references of the original publications and further major research, we refer to the review paper [3].

We note that the PDE problems related to Financial Mathematics are typically Cauchy problems, and initial-boundary value problems arise only after a localization procedure, for the purpose of the numerical schemes' implementation. Therefore, we do not find in most of these problems the complex domain geometries which are one important reason to favour other deterministic numerical methods (e.g., finite-element methods).

Most studies concerning the numerical approximation of PDE problems in Finance consider the particular case where the PDE coefficients are constant (see, e.g., [1, 2, 7, 24]). This occurs, namely, in option pricing under the Black-Scholes model (in one or several dimensions), when the asset appreciation rate and volatility are taken constant. The simpler PDE, with constant coefficients, is obtained after a standard change of variables (see, e.g., [13] for the one-dimensional case, and [8] for the mutidimensional case).

Some other studies develop approximation procedures for more complex models, but restricting the analysis to the case of one spatial dimension (see, e.g., [4, 18]). Although not directly concerning the financial application, we refer also to [5], where convection-dominated PDE problems, in one spatial dimension, with space-dependent coefficients (the first and zero order coefficients taken bounded), are numerically approximated with finite-difference methods.

In [16], a space-time adaptive finite-difference method is developed for the approximation of a multidimensional PDE problem, corresponding to a version of Black-Scholes model where the vector of asset appreciation rates and volatility matrix are taken variable but only with respect to the time variable. The difficulty coming from the unboundedness of the PDE coefficients is not considered, as the discretization is made after a spatial domain truncation.

With the present article, we aim to provide a systematic study of the numerical approximation of the general second-order parabolic problem (1.1), with unbounded coefficients. The study is pursued in the framework of the variational approach, imposing weak regularity over the operator's coefficients and the data f and g. Also, this is the appropriate framework for a future investigation of the correspondent degenerate case.

We summarize the article's content. In Section 2, we establish some well-known facts on the solvability of linear PDEs under a general framework, and introduce a suitable class of weighted Sobolev spaces. In Section 3, we discretize in space problem (1.1), with the use of finite-difference schemes. We set a discrete framework and, by showing that it is a particular case of the general framework presented in the previous Section, we deduce an existence and uniqueness result for the solution of the discretized problem. In Section 4, we prove that the solution of the discretized problem approximates the solution of the continuous problem (1.1), and compute the rate of convergence. In Sections 5 and 6, the approximation in time is considered for evolution equations in abstract spaces, making use of implicit and explicit schemes, respectively. In Section 7, we estimate the rate of convergence for the approximation in space and time. In Section 8, we make some final comments.

2. Preliminaries and classical results

We establish some facts on the solvability of PDEs under a general framework.

Let V be a reflexive separable Banach space embedded continuously and densely into a Hilbert space H with inner product $(\ ,\ )$. Then H^*, the dual space of H, is also continuously and densely embedded into V^*, the dual of V. Let us use the notation $\langle\ ,\ \rangle$ for the duality. Let H^* be identified with H in the usual way, by the help of the inner product. Then we have the so called normal triple $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$, with continuous and dense embeddings.

Let us consider the Cauchy problem for an evolution equation

$$L(t)u(t) - \frac{\partial u(t)}{\partial t} + f(t) = 0, \quad u(0) = g, \tag{2.1}$$

where $L(t)$ and $\partial/\partial t$ are linear operators from V to V^*, $f(t) \in V^*$, for every $t \in [0,T]$ with $T \in (0,\infty)$, and $g \in H$.

We assume that the operator $L(t)$ is continuous and satisfies a coercivity condition, and impose some regularity over the free data f and g:

Assumption 1. There exist constants $\lambda > 0$, K, M and N such that

(1) $\langle L(t)v, v\rangle + \lambda |v|_V^2 \leq K|v|_H^2, \quad \forall v \in V$ and $\forall t \in [0,T]$;
(2) $|L(t)v|_{V^*} \leq M|v|_V, \quad \forall v \in V$ and $\forall t \in [0,T]$;
(3) $\int_0^T |f(t)|_{V^*}^2 dt \leq N$ and $|g|_H \leq N$.

We define the generalized solution of problem (2.1).

Definition 1. We say that $u \in C([0,T];H)$ is a generalized solution of (2.1) on $[0,T]$ if

(1) $u \in L^2([0,T];V)$;
(2) $(u(t), v) = (g, v) + \int_0^t \langle L(s)u(s), v\rangle ds + \int_0^t \langle f(s), v\rangle ds$
holds for every $t \in [0,T]$, $v \in V$.

Notation. Let W be a Banach space. We denote by $C([0,T];W)$ the space of continuous W-valued functions on $[0,T]$. The notation $L^2([0,T];W)$ stands for the space of L^2 W-valued functions on $[0,T]$.

Under Assumption 1, problem (2.1) has a unique generalized solution. The following well-known result is a special case of a more general one proved in [14] for nonlinear evolution equations.

Theorem 1. *Under (1)–(3) in Assumption 1, problem (2.1) has a unique generalized solution on* $[0,T]$*. Moreover*

$$\sup_{t\in[0,T]} |u(t)|_H^2 + \int_0^T |u(t)|_V^2 dt \leq N\Big(|g|_H^2 + \int_0^T |f(t)|_{V^*}^2 dt\Big),$$

where N is a constant.

Let us now consider the particular problem

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ in } \mathbb{R}^d, \tag{2.2}$$

where L is the second-order operator with real coefficients

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c(t,x), \tag{2.3}$$

$Q = [0,T] \times \mathbb{R}^d$, with $T \in (0,\infty)$, and f and g are given functions. We allow the growth, in the space variables, of the coefficients $a^{ij}(t,x)$ and $b^i(t,x)$, $i,j = 1,\dots,d$, and of the free data $f(t,x)$ and $g(x)$.

To set the framework for problem (2.2), we introduce a suitable class of weighted Sobolev spaces.[3]

Let U be a domain in $\mathbb{R}^d$, i.e., an open subset of $\mathbb{R}^d$. Let $r > 0$, $\rho > 0$ be smooth functions in U and $m \geq 0$ an integer. The weighted Sobolev space $W^{m,2}(r,\rho)(U)$ is the normed linear space of locally integrable functions $v : U \to \mathbb{R}$ such that for each multi-index α, with $|\alpha| \leq m$, $D^\alpha v$ exists in the weak sense, and $|v|_{W^{m,2}(r,\rho)(U)} := (\sum_{|\alpha|\leq m} \int_U r^2 |\rho^{|\alpha|} D^\alpha v|^2 dx)^{1/2}$ is finite. The space $W^{m,2}(r,\rho)(U)$ is complete. Endowed with the inner product $(v,w)_{W^{m,2}(r,\rho)(U)} := \sum_{|\alpha|\leq m} \int_U r^2 \rho^{2|\alpha|} D^\alpha v D^\alpha w \, dx$, for all $v, w \in W^{m,2}(r,\rho)(U)$, which generates the norm, $W^{m,2}(r,\rho)(U)$ is a Hilbert space.

Remark 1. Setting the weight functions $r = \rho = 1$, for all $x \in U$, we obtain the particular case of the Sobolev spaces $W^{m,2}(U)$.

Notation. In the sequel, when $U = \mathbb{R}^d$ we drop the argument in the function space notation. For instance, we denote $W^{m,2}(r,\rho)(\mathbb{R}^d) =: W^{m,2}(r,\rho)$.

We make some assumptions on the behaviour of the weight functions r and ρ (see [9]).

Assumption 2. Let $m \geq 0$ be an integer and $r > 0$ and $\rho > 0$ smooth functions on $\mathbb{R}^d$. There exists a constant K such that,

(1) $|D^\alpha \rho| \leq K \rho^{1-|\alpha|}$, for all multi-indexes α such that $|\alpha| \leq m-1$ if $m \geq 2$;
(2) $|D^\alpha r| \leq K \dfrac{r}{\rho^{|\alpha|}}$, for all multi-indexes α such that $|\alpha| \leq m$.

Example 1. The following functions (taken from [21]) satisfy Assumption 2:

(1) $r(x) = (1+|x|^2)^\beta$, $\beta \in \mathbb{R}$; $\quad \rho(x) = (1+|x|^2)^\gamma$, $\gamma \leq \frac{1}{2}$;
(2) $r(x) = \exp(\pm(1+|x|^2)^\beta)$, $0 \leq \beta \leq \frac{1}{2}$; $\quad \rho(x) = (1+|x|^2)^\gamma$, $\gamma \leq \frac{1}{2} - \beta$;
(3) $r(x) = (1+|x|^2)^\beta$, $\beta \in \mathbb{R}$; $\quad \rho(x) = \ln^\gamma(2+|x|^2)$, $\gamma \in \mathbb{R}$;
(4) $r(x) = (1+|x|^2)^\beta \ln^\mu(2+|x|^2)$, $\beta \geq 0$, $\mu \geq 0$; $\quad \rho(x) = (1+|x|^2)^\gamma$, $\gamma \leq \frac{1}{2}$;
(5) $r(x) = (1+|x|^2)^\beta \ln^\mu(2+|x|^2)$, $\beta \geq 0$, $\mu \geq 0$; $\quad \rho(x) = \ln^\gamma(2+|x|^2)$, $\gamma \geq 0$;
(6) $\rho(x) = \exp(-(1+|x|^2)^\gamma)$, $\gamma \geq 0$; $\quad$ each weight function $r(x)$ in examples $(1)-(5)$.

Now, we switch point of view and consider the functions $w : Q \to \mathbb{R}$ as functions on $[0,T]$ with values in $\mathbb{R}^\infty$ such that, for all $t \in [0,T]$, $w(t) := \{w(t,x) : x \in \mathbb{R}^d\}$.

We impose a coercivity condition over the operator (2.3), and make some assumptions on the growth and regularity of the operator's coefficients and also on the regularity of the free data f and g (see [9]):

[3] We refer to [9] for a complete description of this class of spaces.

Assumption 3. Let $r > 0$ and $\rho > 0$ be smooth functions on $\mathbb{R}^d$ and $m \geq 0$ an integer. We assume that

(1) There exists a constant $\lambda > 0$ such that $\sum_{i,j=1}^d a^{ij}(t,x)\xi^i\xi^j \geq \lambda\rho^2(x)\sum_{i=1}^d |\xi^i|^2$, for all $t \geq 0$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$;
(2) The coefficients in L and their derivatives in x up to the order m are measurable functions in $[0,T] \times \mathbb{R}^d$ such that
$$|D_x^\alpha a^{ij}| \leq K\rho^{2-|\alpha|} \ \forall |\alpha| \leq m \vee 1, \ |D_x^\alpha b^i| \leq K\rho^{1-|\alpha|}, \ |D_x^\alpha c| \leq K \ \forall |\alpha| \leq m,$$
for any $t \in [0,T]$, $x \in \mathbb{R}^d$, with K a constant and D_x^α denoting the α^{th} partial derivative operator with respect to x;
(3) $f \in L^2([0,T]; W^{m-1,2}(r,\rho))$ and $g \in W^{m,2}(r,\rho)$.

Remark 2. For $m=0$ we use the notation $W^{m-1,2}(r,\rho) = W^{-1,2}(r,\rho) := (W^{1,2}(r,\rho))^*$, where $(W^{1,2}(r,\rho))^*$ is the dual of $W^{1,2}(r,\rho)$.

We define the generalized solution of problem (2.2).

Definition 2. We say that $u \in C([0,T]; W^{0,2}(r,\rho))$ is a generalized solution of (2.2) on $[0,T]$ if

(1) $u \in L^2([0,T]; W^{1,2}(r,\rho))$;
(2) For every $t \in [0,T]$
$$\begin{aligned}(u(t),\varphi) = (g,\varphi) + \int_0^t \big\{ &-(a^{ij}(s)D_{x^i}u(s), D_{x^j}\varphi) \\ &+ (b^i(s)D_{x^i}u(s) - D_{x^j}a^{ij}(s)D_{x^i}u(s), \varphi) \\ &+ (c(s)u(s),\varphi) + \langle f(s),\varphi\rangle \big\} ds\end{aligned}$$
holds for all $\varphi \in C_0^\infty$.

Notation. The notation (,) in the above definition stands for the inner product in $W^{0,2}(r,\rho)$. C_0^∞ denotes the set of all infinitely differentiable functions on $\mathbb{R}^d$ with compact support.

Remark 3. Note that, alternatively to the infinite differentiability of φ in (2) it can be required that $\varphi \in W^{1,2}(r,\rho)$.

Finally, we state a result on the existence and uniqueness of the solution of problem (2.2). This result can be obtained from the general result in abstract spaces (Theorem 1) by using the suitable triple of spaces (see [9]).

Theorem 2. *Under (1)–(2) in Assumption 2, with $m+1$ in place of m, and (1)–(3) in Assumption 3, problem (2.2) admits a unique generalized solution u on $[0,T]$. Moreover $u \in C([0,T]; W^{m,2}(r,\rho)) \cap L^2([0,T]; W^{m+1,2}(r,\rho))$ and*
$$\sup_{0\leq t\leq T} |u(t)|^2_{W^{m,2}(r,\rho)} + \int_0^T |u(t)|^2_{W^{m+1,2}(r,\rho)} dt \leq N\Big(|g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)} dt \Big),$$
with N a constant.

3. Numerical approximation in space: the discrete framework

In this Section we proceed to the discretization of problem (2.2) in the space-variables. We set a suitable discrete framework with the use of a finite-difference scheme and, by showing that discretized problem can be cast into the general problem (2.1), we prove an existence and uniqueness result for the discrete problem's generalized solution.

We define the h-grid on $\mathbb{R}^d$, with $h \in (0,1]$

$$Z_h^d = \Big\{ x \in \mathbb{R}^d : x = h \sum_{i=1}^{d} e_i n_i, \quad n_i = 0, \pm 1, \pm 2, \dots \Big\}.$$

Denote

$$\partial_i^+ u = \partial_i^+ u(t,x) = h^{-1}(u(t, x + he_i) - u(t,x))$$

and

$$\partial_i^- u = \partial_i^- u(t,x) = h^{-1}(u(t,x) - u(t, x - he_i)),$$

for every $x \in Z_h^d$, the forward and backward discrete differences in space, respectively. Define the discrete operator

$$L_h(t,x) = a^{ij}(t,x)\partial_j^- \partial_i^+ + b^i(t,x)\partial_i^+ + c\,(t,x).$$

We consider the discrete problem

$$L_h u - u_t + f_h = 0 \ \text{ in } \ Q(h), \quad u(0,x) = g_h(x) \ \text{ in } \ Z_h^d, \tag{3.1}$$

where $Q(h) = [0,T] \times Z_h^d$, with $T \in (0, \infty)$, and f_h and g_h are functions such that $f_h : Q(h) \to \mathbb{R}$ and $g_h : Z_h^d \to \mathbb{R}$.

Consider functions $v : Z_h^d \to \mathbb{R}$. We introduce the discrete version of the weighted Sobolev space $W^{0,2}(r, \rho)$:

$$l^{0,2}(r) = \{ v : Z_h^d \to \mathbb{R} : |v|_{l^{0,2}(r)} < \infty \},$$

where the norm $|v|_{l^{0,2}(r)}$ is defined by

$$|v|_{l^{0,2}(r)} = \Big(\sum_{x \in Z_h^d} r^2(x) |v(x)|^2 h^d \Big)^{1/2}.$$

Define the inner product

$$(v, w)_{l^{0,2}(r)} = \sum_{x \in Z_h^d} r^2(x) v(x) w(x) h^d,$$

for any $v, w \in l^{0,2}(r)$, which induces the above norm.

We show that space $l^{0,2}(r)$ has a good structure.

Proposition 1. *The function space $l^{0,2}(r)$ is a Hilbert space.*

Proof. To prove that $l^{0,2}(r)$ is a Hilbert space we have to prove that the inner product space $l^{0,2}(r)$ is complete, i.e., that any Cauchy sequence in $l^{0,2}(r)$ is convergent in the space norm.

Let (v_n) be a Cauchy sequence in $l^{0,2}(r)$, that is, for all $\varepsilon > 0$ exists N such that for $m, n > N$

$$|v_m - v_n|_{l^{0,2}(r)} = \Big(\sum_{x \in Z_h^d} r^2(x) |v_m(x) - v_n(x)|^2 h^d \Big)^{1/2} < \varepsilon. \tag{3.2}$$

Then, for every $x \in Z_h^d$ we have

$$r^2(x) |v_m(x) - v_n(x)|^2 h^d < \varepsilon^2, \qquad \text{for } \ m, n > N. \tag{3.3}$$

Let us fix $x = x_0$. From (3.3) we see that $(v_1(x_0), v_2(x_0), \ldots)$ is a Cauchy sequence of real numbers, therefore convergent. Write $v_m(x_0) \to v(x_0)$. Using these limits, we define $v = v(x)$, for each $x \in Z_h^d$.

Let B be a ball in Z_h^d. From (3.2) for $m, n > N$

$$\sum_{x \in B} r^2(x)|v_m(x) - v_n(x)|^2 h^d < \varepsilon^2.$$

Letting $n \to \infty$, for $m > N$

$$\sum_{x \in B} r^2(x)|v_m(x) - v(x)|^2 h^d \le \varepsilon^2.$$

Letting now the diameter of B go to ∞, for $m > N$

$$\sum_{x \in Z_h^d} r^2(x)|v_m(x) - v(x)|^2 h^d \le \varepsilon^2. \tag{3.4}$$

Inequality (3.4) implies that $v_m - v \in l^{0,2}(r)$. As $v_m \in l^{0,2}(r)$, it follows, owing to Minkowski inequality for sums, that

$$v = v_m + (v - v_m) \in l^{0,2}(r).$$

Finally, (3.4) also implies that $v_m \to v$ and the result is proved. □

For functions $v : Z_h^d \to \mathbb{R}$, we introduce also the discrete version of the weighted Sobolev space $W^{1,2}(r, \rho)$:

$$l^{1,2}(r, \rho) = \{v : Z_h^d \to \mathbb{R} : \ |v|_{l^{1,2}(r,\rho)} < \infty\},$$

with the norm $|v|_{l^{1,2}(r,\rho)}$ defined by

$$|v|^2_{l^{1,2}(r,\rho)} = |v|^2_{l^{0,2}(r)} + \sum_{i=1}^{d} |\rho\, \partial_i^+ v|^2_{l^{0,2}(r)}.$$

We endow $l^{1,2}(r, \rho)$ with the inner product, inducing the above norm,

$$(v, w)_{l^{1,2}(r,\rho)} = (v, w)_{l^{0,2}(r)} + \sum_{i=1}^{d} (\rho\, \partial_i^+ v, \rho\, \partial_i^+ w)_{l^{0,2}(r)},$$

for v, w any functions in $l^{1,2}(r, \rho)$.

To prove that the discrete framework we set is a particular case of the general framework considered in Section 2, we begin by checking that $l^{1,2}(r, \rho)$ is a reflexive and separable Banach space, continuously and densely embedded into the Hilbert space $l^{0,2}(r)$.

Following the same steps as in the proof of Proposition 1, it can be easily proved that $l^{1,2}(r, \rho)$ is a complete inner product space. Therefore $l^{1,2}(r, \rho)$ is reflexive. We prove next that $l^{1,2}(r, \rho)$ is separable.

Proposition 2. *The function space $l^{1,2}(r, \rho)$ is separable.*

Proof. We have to prove that $l^{1,2}(r, \rho)$ has a countable dense subset.

Let us consider the set $S = B \cup \{x + he_i : x \in B,\ i = 1, 2, \ldots, d\}$, with B a ball in Z_h^d. Consider the set of all functions $w(x) \in l^{1,2}(r, \rho)$ taking rational values if $x \in S$ and vanishing outside S, and denote it by l. The set l is countable.

Let v be an arbitrary function in $l^{1,2}(r,\rho)$. For any given $\varepsilon > 0$, we can choose $w \in l$ such that

$$
\begin{aligned}
&\sum_{x\in B} r^2(x)|v(x)-w(x)|^2h^d + \sum_{i=1}^{d}\sum_{x\in B} r^2(x)|\rho(x)\partial_i^+(v(x)-w(x))|^2h^d \\
&= \sum_{x\in B} r^2(x)|v(x)-w(x)|^2h^d \\
&+\sum_{i=1}^{d}\sum_{x\in B} r^2(x)|\rho(x)h^{-1}(v(x+he_i)-w(x+he_i)-(v(x)-w(x)))|^2h^d \\
&\le \sum_{x\in B} M^2|v(x)-w(x)|^2h^d + 2\sum_{i=1}^{d}\sum_{x\in B} M^2N^2|v(x+he_i)-w(x+he_i)|^2h^{d-2} \\
&+2\sum_{i=1}^{d}\sum_{x\in B} M^2N^2|v(x)-w(x)|^2h^{d-2} < \frac{\varepsilon^2}{2},
\end{aligned}
\tag{3.5}
$$

with M and N the suprema in B of r and ρ, respectively.

As $|v|^2_{l^{1,2}(r,\rho)}$ is an absolutely convergent series, for any given $\varepsilon > 0$ we can choose the diameter of B such that

$$
\sum_{x\notin B} r^2(x)|v(x)|^2h^d + \sum_{i=1}^{d}\sum_{x\notin B} r^2(x)|\rho(x)\partial_i^+ v(x)|^2h^d < \frac{\varepsilon^2}{2}. \tag{3.6}
$$

From (3.5) and (3.6) we obtain

$$
|v-w|_{l^{1,2}(r,\rho)} < \varepsilon,
$$

and the result is proved. □

We now check that $l^{1,2}(r,\rho)$ is continuously and densely embedded in $l^{0,2}(r)$. The continuity follows immediately from

$$
|v|_{l^{0,2}(r)} \le |v|_{l^{1,2}(r,\rho)}, \quad \text{for all } \ v \in l^{1,2}(r,\rho).
$$

For the denseness, we prove the following result:

Proposition 3. *The function space $l^{1,2}(r,\rho)$ is densely embedded in $l^{0,2}(r)$.*

Proof. We want to prove that $\overline{l^{1,2}(r,\rho)} = l^{0,2}(r)$. Let us take an arbitrary function $v \in l^{0,2}(r)$. Let B be a ball in Z_h^d. We consider the function w such that

$$
w(x) = \begin{cases} v(x), & x \in B \\ 0, & \text{otherwise.} \end{cases}
$$

This function belongs obviously to $l^{1,2}(r,\rho)$. Furthermore, for any given $\varepsilon > 0$, we have

$$
|v-w|_{l^{0,2}(r)} < \varepsilon,
$$

if the diameter of B is chosen sufficiently large. The result is proved. □

Now, we switch our viewpoint and consider the functions $w : Q(h) \to \mathbb{R}$ as functions in $[0,T]$ with values in $\mathbb{R}^\infty$, defined by $w(t) = \{w(t,x) : x \in Z_h^d\}$, for all

$t \in [0,T]$. For these functions, we consider the space $C([0,T]; l^{0,2}(r))$ of continuous $l^{0,2}(r)$-valued functions on $[0,T]$ and the spaces

$$L^2([0,T]; l^{m,2}(r,\rho)) = \Big\{ w : [0,T] \to l^{m,2}(r,\rho) : \int_0^T |w(t)|^2_{l^{m,2}(r,\rho)}\, dt < \infty \Big\},$$

with $m = 0, 1$.

Remark 4. Clearly, if $u \in C([0,T]; l^{0,2}(r))$ then $\sup_{t\in[0,T]} |u(t)|_{l^{0,2}(r)} < \infty$.

We make some assumptions over the regularity of the data f_h and g_h in (3.1).

Assumption 4. Let $r > 0$ be a smooth function on $\mathbb{R}^d$.

(1) $f_h \in L^2([0,T]; l^{0,2}(r))$;
(2) $g_h \in l^{0,2}(r)$.

Remark 5. In the above Assumption 4, (1) can be replaced for the weaker assumption $f_h \in L^2([0,T]; (l^{1,2}(r,\rho))^*)$, where $(l^{1,2}(r,\rho))^*$ denotes the dual space of $l^{1,2}(r,\rho)$.

Remark 6. We note that $|\partial_i^+ a^{ij}| \le K\rho$ can be obtained from (2) in Assumption 3. In fact,

$$|\partial_i^+ a^{ij}(t,x)| = |h^{-1}(a^{ij}(t,x+he_i) - a^{ij}(t,x))| \le \Big|\frac{\partial}{\partial x^i}\, a^{ij}(t,x+\tau e_i)\Big|,$$

for some τ such that $0 < \tau < h$. Thus $|(\partial/\partial x^i)\, a^{ij}| \le K\rho$ implies $|\partial_i^+ a^{ij}| \le K\rho$.

We define the generalized solution of problem (3.1).

Definition 3. We say that $u \in C([0,T]; l^{0,2}(r)) \cap L^2([0,T]; l^{1,2}(r,\rho))$ is a generalized solution of (3.1) if

$$(u(t),\varphi) = (g_h,\varphi) + \int_0^t \big\{ -(a^{ij}(s)\partial_i^+ u(s), \partial_j^+\varphi) + (b^i(s)\partial_i^+ u(s) - \partial_j^+ a^{ij}(s)\partial_i^+ u(s), \varphi) + (c(s)u(s),\varphi) + \langle f_h(s),\varphi\rangle \big\} ds$$

holds for every $t \in [0,T]$, $\varphi \in l^{1,2}(r,\rho)$.

Notation. In the above definition, (,) denotes the inner product in $l^{0,2}(r)$. We keep this convention for the remaining of present Section.

Next, we prove an existence and uniqueness result for the solution of the discrete problem (3.1), computing, in addition, an estimate for the solution. With this result, we show that the numerical scheme is stable, i.e., informally, that the discrete problem's solution remains bounded independently of the space-step h. The result is obtained as a consequence of Theorem 1, remaining only to show that, within the discrete framework we constructed, (1) − (2) in Assumption 1 hold.

Theorem 3. *Under (1)−(2) in Assumption 3 and (1)−(2) in Assumption 4, problem (3.1) has a unique generalized solution u in $[0,T]$. Moreover*

$$\sup_{0\le t\le T} |u(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t)|^2_{l^{1,2}(r,\rho)}\, dt \;\le\; N\, \Big(|g_h|^2_{l^{0,2}(r)} + \int_0^T |f_h(t)|^2_{l^{0,2}(r)}\, dt \Big),$$

with N a constant independent of h.

Proof. Let $L_h(s) : l^{1,2}(r,\rho) \to (l^{1,2}(r,\rho))^*$ for all $s \in [0,T]$. We define for all $s \in [0,T]$, $\varphi, \psi \in l^{1,2}(r,\rho)$

$$\langle L_h(s)\psi,\varphi\rangle := -\,(a^{ij}(s)\partial_i^+\psi, \partial_j^+\varphi) + (b^i(s)\partial_i^+\psi - \partial_j^+ a^{ij}(s)\partial_i^+\psi, \varphi) + (c(s)\psi,\varphi).$$

It suffices to prove that the following estimates hold

(1) $\exists K, \lambda > 0$ constants : $\langle L_h(s)\psi,\psi\rangle \le K|\psi|_{l^{0,2}(r)} - \lambda|\psi|_{l^{1,2}(r,\rho)}$
(2) $\exists K$ constant : $|\langle L_h(s)\psi,\varphi\rangle| \le K|\psi|_{l^{1,2}(r,\rho)} \cdot |\varphi|_{l^{1,2}(r,\rho)}$

for all $s \in [0,T]$, $\varphi, \psi \in l^{1,2}(r,\rho)$.

For the first property, owing to (1) and (2) in Assumption 3, we have

$$\begin{aligned}\langle L_h(s)\psi,\psi\rangle &= -\sum_{i,j}\sum_x r^2 a^{ij}(s)\partial_i^+\psi\,\partial_j^+\psi\,h^d \\ &\quad + \sum_i\sum_x r^2(b^i(s) - \partial_j^+ a^{ij}(s))\partial_i^+\psi\,\psi\,h^d + \sum_x r^2 c(s)\psi\,\psi\,h^d \\ &\le -\lambda\sum_i\sum_x r^2|\rho\partial_i^+\psi|^2h^d + 2K\sum_i\sum_x r^2\rho|\partial_i^+\psi\,\psi|h^d + K\sum_x r^2|\psi|^2h^d \\ &= -\lambda\sum_i|\rho\partial_i^+\psi|^2_{l^{0,2}(r)} + 2K\sum_i\sum_x r^2\rho|\partial_i^+\psi\,\psi|h^d + K|\psi|^2_{l^{0,2}(r)},\end{aligned} \tag{3.7}$$

where the variable $x \in Z_h^d$ is omitted, $\sum_x$ denotes the summation over Z_h^d and $\sum_i$, $\sum_j$ the summation over $\{1,2,\dots,d\}$. Applying Cauchy's inequality to the second term in estimate (3.7), we obtain

$$\begin{aligned}&\langle L_h(s)\psi,\psi\rangle \\ &\le -\lambda\sum_i|\rho\partial_i^+\psi|^2_{l^{0,2}(r)} + \varepsilon K\sum_i\sum_x r^2|\rho\partial_i^+\psi|^2h^d + \frac{K}{\varepsilon}\sum_i\sum_x r^2|\psi|^2h^d + K|\psi|^2_{l^{0,2}(r)} \\ &= -\lambda\sum_i|\rho\partial_i^+\psi|^2_{l^{0,2}(r)} - \lambda|\psi|^2_{l^{0,2}(r)} + \varepsilon K\sum_i|\rho\partial_i^+\psi|^2_{l^{0,2}(r)} + \frac{K}{\varepsilon}|\psi|^2_{l^{0,2}(r)} + (K+\lambda)|\psi|^2_{l^{0,2}(r)} \\ &\le -\lambda|\psi|^2_{l^{1,2}(r,\rho)} + K|\psi|^2_{l^{0,2}(r)},\end{aligned}$$

with $\lambda > 0$, K constants, by taking ε sufficiently small, and the first property is proved.

The second property follows from (2) in Assumption 3 and Cauchy-Schwarz inequality

$$\begin{aligned}&\big|\langle L_h(s)\psi,\,\varphi\rangle\big| \\ &= \big|-\sum_{i,j}\sum_x r^2a^{ij}(s)\partial_i^+\psi\,\partial_j^+\varphi\,h^d + \sum_i\sum_x r^2b^i(s)\partial_i^+\psi\,\varphi\,h^d \\ &\quad - \sum_{i,j}\sum_x r^2\partial_j^+a^{ij}(s)\partial_i^+\psi\,\varphi\,h^d + \sum_x r^2c(s)\psi\,\varphi\,h^d\big| \\ &\le K\sum_{i,j}\sum_x r^2|\rho^2\partial_i^+\psi\,\partial_j^+\varphi|\,h^d + K\sum_i\sum_x r^2|\rho\partial_i^+\psi\,\varphi|\,h^d + K\sum_x r^2|\psi\,\varphi|h^d \\ &\le K\sum_i|\rho\partial_i^+\psi|_{l^{0,2}(r)}\sum_j|\rho\partial_j^+\varphi|_{l^{0,2}(r)} + K\sum_i|\rho\partial_i^+\psi|_{l^{0,2}(r)}|\varphi|_{l^{0,2}(r)} + K|\psi|_{l^{0,2}(r)}|\varphi|_{l^{0,2}(r)} \\ &\le K|\psi|_{l^{1,2}(r,\rho)}\cdot|\varphi|_{l^{1,2}(r,\rho)},\end{aligned}$$

where the same writing conventions are kept.

Owing to Theorem 1 the result follows. □

4. Numerical approximation in space: approximation results

In this Section, we study the approximation properties of the numerical scheme (3.1). We begin by investigating the consistency of the numerical scheme, and prove that the discrete differences approximate the partial derivatives (with accuracy of order 1). The result is obtained by using a Sobolev inequality, under stronger regularity assumptions, and imposing that the weights ρ are bounded from below by a positive constant. In practice, the latter restriction amounts to assume that the weights ρ are increasing functions of $|x|$, which is precisely the case we are studying.

Theorem 4. *Let $r > 0$ and $\rho > 0$ be functions on $\mathbb{R}^d$, and m an integer strictly greater than $d/2$. Assume that (1)–(2) in Assumption 2 are satisfied and that, additionally, $\rho(x) \geq C$ on $\mathbb{R}^d$, with $C > 0$ a constant. Let $u(t) \in W^{m+2,2}(r,\rho)$, $v(t) \in W^{m+3,2}(r,\rho)$, for all $t \in [0,T]$. Then there exists a constant N independent of h such that*

(1) $$\sum_{x\in Z_h^d} r^2(x)|u_{x^i}(t,x) - \partial_i^+ u(t,x)|^2 \rho^2(x) h^d \leq h^2 N |u(t)|^2_{W^{m+2,2}(r,\rho)},$$

(2) $$\sum_{x\in Z_h^d} r^2(x)|v_{x^i x^j}(t,x) - \partial_j^- \partial_i^+ v(t,x)|^2 \rho^4(x) h^d \leq h^2 N |v(t)|^2_{W^{m+3,2}(r,\rho)},$$

for all $t \in [0,T]$.

Remark 7. The following remarks will be used in the proof of the theorem:

(1) Under the conditions of the theorem, function $u(t)$ (function $v(t)$) has a modification in x which is continuously differentiable in x up to the order 2 (up to the order 3), and the derivatives equal the weak derivatives, for every $t \in [0,T]$. This can be proved by Sobolev's embedding of $W^{m,2}(B)$ into $C^n(\overline{B})$, for balls B in $\mathbb{R}^d$, if $m > \frac{d}{2} + n$, and using Morrey's inequality (see, e.g, [6, 14, 15]). We consider these modifications in the theorem's proof.

(2) Note that if U, V are open subsets of $\mathbb{R}^d$ with $V \subset U$ and $w \in W^{m,2}(U)$ then $w \in W^{m,2}(V)$. Also, if $w \in W^{m,2}(U)$ and $\zeta \in C_0^\infty(U)$ then $\zeta \in W^{m,2}(U)$ and $\zeta w \in W^{m,2}(U)$ (see, e.g, [6, 14, 15]).

Proof. (Theorem 4) Let us prove (1). We will define a suitable geometric setting, and then obtain an estimate for

$$r^2(x)|u_{x^i}(t,x) - \partial_i^+ u(t,x)|^2 \rho^2(x),$$

with $x \in Z_h^d$, using Sobolev's inequality on a fixed ball.

Let us consider d-cells

$$R_h = \{(x^1, x^2, \ldots, x^d) \in \mathbb{R}^d : \ \ x_h^i < x^i < x_h^i + h, \ \ i = 1, 2, \ldots, d\},$$

with $x_h = (x_h^1, x_h^2, \ldots, x_h^d) \in Z_h^d$ fixed. Consider the particular d-cell where $h = 1$ and $x_h = x_1 = (0, 0, \ldots, 0)$ and denote it R_1^0. Now, take open balls B_h such that $B_h \supset R_h$, with the vertices $\{x_h^i, x_h^i + h, \ i = 1, 2, \ldots, d\}$ laying on the limiting sphere. Denote B_1^0 the ball containing R_1^0.

For every $x_h \in Z_h^d$, taking in mind (1) in Remark 7, we have by the mean-value theorem

$$\partial_i^+ u(t, x_h) = h^{-1}(u(t, x_h + he_i) - u(t, x_h)) = u_{x^i}(t, x_h + \theta h e_i)$$

and

$$u_{x^i}(t,x_h)-\partial_i^+u(t,x_h)=u_{x^i}(t,x_h)-u_{x^i}(t,x_h+\theta he_i)\leq h\,u_{x^ix^i}(t,x_h+\theta' he_i), \tag{4.1}$$

for some $0<\theta'<\theta<1$.

Clearly

$$|u_{x^ix^i}(t,x_h+\theta' he_i)|\leq \sup_{x\in R_h}|u_{x^ix^i}(t,x)|, \tag{4.2}$$

and then, by (4.1) and (4.2),

$$|u_{x^i}(t,x_h)-\partial_i^+u(t,x_h)|^2\leq h^2\sup_{x\in R_h}|u_{x^ix^i}(t,x)|^2. \tag{4.3}$$

We change variable in order to have the supremum calculated over the fixed d-cell R_1^0:

$$\sup_{x\in R_h}|u_{x^ix^i}(t,x)|=\sup_{x\in R_1^0}|u_{x^ix^i}(t,x_h+hx)|. \tag{4.4}$$

As

$$\sup_{x\in R_1^0}|u_{x^ix^i}(t,x_h+hx)|^2\leq \sup_{x\in B_1^0}|u_{x^ix^i}(t,x_h+hx)|^2, \tag{4.5}$$

from (4.3) – (4.5) we immediately obtain

$$\begin{aligned}&r^2(x_h)|u_{x^i}(t,x_h)-\partial_i^+u(t,x_h)|^2\rho^2(x_h)\\&\leq h^2\sup_{x\in R_1^0}\big(r^2(x_h+hx)|u_{x^ix^i}(t,x_h+hx)|^2\rho^2(x_h+hx)\big)\\&\leq h^2\sup_{x\in B_1^0}\big(r^2(x_h+hx)|u_{x^ix^i}(t,x_h+hx)|^2\rho^2(x_h+hx)\big).\end{aligned} \tag{4.6}$$

Taking in mind (2) in Remark 7, we have for $m>d/2$ by Sobolev's inequality

$$\begin{aligned}&\sup_{x\in B_1^0}|r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)|^2\\&\leq N\sum_{|\alpha|\leq m}\int_{B_1^0}\big|D_x^\alpha\big(r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)\big)\big|^2dx,\end{aligned} \tag{4.7}$$

with N a constant independent of h.

Observe that, owing to Leibniz' formula,

$$\begin{aligned}|D_x^\alpha(ru_{x^ix^i}\rho)|&=\Big|\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}D^\beta(r\rho)D_x^{\alpha-\beta}u_{x^ix^i}\Big|\\&=\Big|\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\Big(\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}D^\gamma rD^{\beta-\gamma}\rho\Big)D_x^{\alpha-\beta}u_{x^ix^i}\Big|,\end{aligned} \tag{4.8}$$

where the arguments of r, ρ and $u_{x^ix^i}$ are omitted.

Also, keeping the same convention, owing to Assumption 2

$$|D^\gamma r|\leq Kr\rho^{-|\gamma|}\quad\text{and}\quad|D^{\beta-\gamma}\rho|\leq K\rho^{1-(|\beta|-|\gamma|)},$$

with K a constant, and then

$$\Big|\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}D^\gamma rD^{\beta-\gamma}\rho\Big|\leq N\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}r\rho^{-|\gamma|}\rho^{1-(|\beta|-|\gamma|)}\leq Nr\rho^{1-|\beta|}, \tag{4.9}$$

with N a constant.

Therefore, by (4.7) − (4.9), we get

$$
\begin{aligned}
&\sup_{x\in B_1^0}|r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)|^2\\
&\le N\sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\int_{B_1^0} r^2(x_h+hx)|\rho^{1-|\beta|}(x_h+hx)|^2|D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)|^2dx.
\end{aligned}
\tag{4.10}
$$

Note also that, using Hölder inequality, owing to the hypotheses over function ρ, the integral in (4.10) can be estimated by

$$
\begin{aligned}
&\int_{B_1^0} r^2(x_h+hx)|\rho^{1-|\beta|}(x_h+hx)|^2 D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)|^2dx\\
&\le N\int_{B_1^0} r^2(x_h+hx)|\rho^{2+(|\alpha|-|\beta|)}(x_h+hx)|^2 D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)|^2dx\\
&\qquad\cdot\sup_{x\in B_1^0}|\rho^{-1-|\alpha|}(x_h+hx)|^2\\
&\le N\int_{B_1^0} r^2(x_h+hx)|\rho^{2+(|\alpha|-|\beta|)}(x_h+hx)|^2 D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)|^2dx.
\end{aligned}
\tag{4.11}
$$

Now, by (4.10) and (4.11),

$$
\begin{aligned}
&\sup_{x\in B_1^0}|r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)|^2\\
&\le N\sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\int_{B_1^0} r^2(x_h+hx)|\rho^{2+(|\alpha|-|\beta|)}(x_h+hx)|D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)|^2dx\\
&\le N\sum_{|\alpha|\le m}\int_{B_1^0} r^2(x_h+hx)|\rho^{2+|\alpha|}(x_h+hx)|D_x^{\alpha}u_{x^ix^i}(t,x_h+hx)|^2dx\\
&\le N\sum_{|\alpha|\le m+2}\int_{B_1^0} r^2(x_h+hx)|\rho^{|\alpha|}(x_h+hx)D_x^{\alpha}u(t,x_h+hx)|^2dx\\
&= N\sum_{|\alpha|\le m+2}\int_{B_h} r^2(x)|\rho^{|\alpha|}(x)D_x^{\alpha}u(t,x)|^2h^{-d}h^{2|\alpha|}dx\\
&\le N\sum_{|\alpha|\le m+2}\int_{B_h} r^2(x)|\rho^{|\alpha|}(x)D_x^{\alpha}u(t,x)|^2h^{-d}dx.
\end{aligned}
\tag{4.12}
$$

Finally, by (4.6) and (4.12), owing to the particular geometry of the framework we have set, we obtain

$$
\begin{aligned}
&\sum_{x\in Z_h^d} r^2(x)|u_{x^i}(t,x)-\partial_i^+u(t,x)|^2\rho^2(x)h^d\\
&\le Nh^2\sum_{|\alpha|\le m+2}\sum_{x_h\in Z_h^d}\int_{B_h(x_h)} r^2(x)|\rho^{|\alpha|}(x)D_x^{\alpha}u(t,x)|^2dx\\
&\le Nh^2\sum_{|\alpha|\le m+2}\sum_{x_h\in Z_h^d}\int_{R_h(x_h)} r^2(x)|\rho^{|\alpha|}(x)D_x^{\alpha}u(t,x)|^2dx\le h^2N|u(t)|^2_{W^{m+2,2}(r,\rho)},
\end{aligned}
$$

where $B_h(x_h) := B_h$, $R_h(x_h) := R_h$, and the proof for (1) is complete. The proof for (2) follows the same steps. □

Finally, owing to the stability and consistency properties of the numerical scheme (Theorems 3 and 4, respectively), we prove the convergence of the discrete problem's solution to the PDE problem's solution, and compute a rate of convergence. The accuracy obtained is of order 1.

Theorem 5. *Let Let $r > 0$ and $\rho > 0$ be functions on $\mathbb{R}^d$, and m an integer strictly greater than $d/2$. Assume that the hypotheses of Theorems 2 and 3 are satisfied. Assume additionally that $\rho(x) \geq C$ on $\mathbb{R}^d$, with $C > 0$ a constant. Denote u the solution of problem (2.2) in Theorem 2 and u_h the solution of problem (3.1) in Theorem 3. Assume also that $u \in L^2([0,T]; W^{m+3,2}(r,\rho))$. Then*

$$\sup_{0\leq t\leq T} |u(t) - u_h(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}(r,\rho)}\, dt$$
$$\leq h^2 N \int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)} dt + N\Big(|g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\, dt\Big),$$

with N a constant independent of h.

Remark 8. Under the conditions of theorem, there are modifications in x such that the data $f(t)$ and g are continuous in x, for every $t \in [0,T]$ (see Remark 7). We will consider these modifications in the proof of the theorem.

Proof. (Theorem 5) From (2.2) and (3.1), we have that $u - u_h$ satisfies the problem

$$\begin{cases} L_h(u-u_h) - \frac{\partial}{\partial t}(u-u_h) + (L-L_h)u + (f-f_h) = 0 & \text{in } Q(h) \\ (u-u_h)(0,x) = (g-g_h)(x) & \text{in } Z_h^d. \end{cases} \tag{4.13}$$

Taking in mind Remark 8, $f - f_h \in L^2([0,T]; l^{0,2}(r))$ and $g - g_h \in l^{0,2}(r)$. With respect to the term $(L - L_h)u$, note that if $u(t) \in W^{m+3,2}(r,\rho)$, for all $t \in [0,T]$,

$$\sum_{x\in Z_h^d} r^2(x)|(L-L_h)(t)u(t)|^2\; h^d$$
$$= \sum_{x\in Z_h^d} r^2(x)\big|a^{ij}(t,x)\big(\frac{\partial^2}{\partial x^i \partial x^j} - \partial_j^- \partial_i^+\big)u(t,x) + b^i(t,x)\big(\frac{\partial}{\partial x^i} - \partial_i^+\big)u(t,x)\big|^2\; h^d < \infty,$$

owing to (2) in Assumption 3 and to Theorem 4. Thus $(L - L_h)(t)u(t) \in l^{0,2}(r)$ for every $t \in [0,T]$, and by continuity in t we have $(L - L_h)u \in L^2([0,T]; l^{0,2}(r))$.

We have shown that problem (4.13) satisfies the hypotheses of Theorem 3, therefore holding the estimate

$$\sup_{0\leq t\leq T} |u(t) - u_h(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}(r,\rho)}\, dt$$
$$\leq N\Big(|g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\, dt + \int_0^T |(L - L_h)u(t)|^2_{l^{0,2}(r)}\, dt\Big).$$

Owing again to (2) in Assumption 3 and to Theorem 4, the result follows. □

Next result is an immediate consequence of Theorem 5.

Corollary 1. *Let the hypotheses of Theorem 5 be satisfied, and denote u the solution of (2.2) in Theorem 2 and u_h the solution of (3.1) in Theorem 3. If there is a constant N independent of h such that*

$$|g-g_h|^2_{l^{0,2}(r)}+\int_0^T |f(t)-f_h(t)|^2_{l^{0,2}(r)}\,dt \le h^2N\Big(|g|^2_{W^{m,2}(r,\rho)}+\int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)}dt\Big),$$

then

$$\begin{aligned}&\sup_{0\le t\le T} |u(t)-u_h(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t)-u_h(t)|^2_{l^{1,2}(r,\rho)}\,dt\\ &\le h^2N\Big(\int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)}dt + |g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)}dt\Big).\end{aligned}$$

5. Numerical approximation in time: implicit scheme

In Sections 3 and 4, we proceeded to the approximation in the space variables of the solution of the second-order parabolic PDE problem (2.2).

We will now study the approximation in time of the solution of the more general problem (2.1) for linear evolution equations, by constructing discrete versions of the general framework we presented in Section 2. This simpler general approach in abstract spaces is powerful enough in order to obtain the desired results. In the present section we investigate the time-discretization of problem (2.1), making use of an implicit finite-difference scheme. The approach using an explicit scheme will be considered in Section 6.

We begin by setting an appropriate discrete framework, and construct a time-discretized version of problem (2.1).

Take a number $T\in(0,\infty)$, a non-negative integer n such that $T/n\in(0,1]$ and define the n-grid on $[0,T]$

$$T_n=\{t\in[0,T]: t=k\varepsilon,\quad k=0,1,\dots,n\}, \tag{5.1}$$

where $\varepsilon := T/n$. Denote $t_k=k\varepsilon$ for $k=0,1,\dots,n$.

For all $z\in V$, we consider the backward discrete difference in time

$$\Delta^- z(t_{j+1})=\varepsilon^{-1}(z(t_{j+1})-z(t_j)),\quad j=0,1,\dots,n-1.$$

Let L_ε, f_ε be some time-discrete versions of L and f, respectively. For all $z\in V$, denote $L_{\varepsilon,j+1}z=L_\varepsilon(t_{j+1})z$, $f_{\varepsilon,j+1}=f_\varepsilon(t_{j+1})$, $j=0,1,\dots,n-1$.

For each $n\ge1$ fixed, we define $v_j=v(t_j)$, $j=0,1,\dots,n$, a vector in V satisfying

$$\Delta^- v_{i+1}=L_{\varepsilon,i+1}v_{i+1}+f_{\varepsilon,i+1}\ \text{ for }\ i=0,1,\dots,n-1,\quad v_0=g. \tag{5.2}$$

Problem (5.2) is a time-discrete version of problem (2.1).

Assumption 5. We assume

(1) $\langle L_{\varepsilon,j+1}v,v\rangle+\lambda|v|^2_V\le K|v|^2_H,\quad \forall v\in V,\quad j=0,1,\dots,n-1$
(2) $|L_{\varepsilon,j+1}v|_{V^*}\le M|v|_V,\quad \forall v\in V,\quad j=0,1,\dots,n-1$
(3) $\sum_{j=0}^{n-1}|f_{\varepsilon,j+1}|^2_{V^*}\varepsilon\le N$ and $|g|_H\le N$,

where λ, K, M and N are the constants in Assumption 1.

Remark 9. Note that as problem (5.2) is a time discretization of problem (2.1) and g denotes the same function in both problems, under Assumption 1 $g\in H$ and $|g|_H\le N$.

Under the above assumptions, we establish the existence and uniqueness of the solution of problem (5.2).

Theorem 6. *Under Assumption 5, for all $n \in \mathbb{N}$ there exists a unique vector $v_0, v_1, \ldots, v_n$ in V satisfying (5.2).*

To prove this result, we consider a well known lemma holding even for a class of nonlinear operators (see [28]).

Lemma 1. *Let $B : V \to V^*$ be a bounded linear operator. Assume there exists $\lambda > 0$ such that $\langle Bv, v\rangle \geq \lambda |v|_V^2$, for all $v \in V$. Then $Bv = v^*$ has a unique solution $v \in V$ for every given $v^* \in V^*$.*

We will now prove Theorem 6.

Proof. (Theorem 6) From (5.2), we have that $(I - \varepsilon\, L_{\varepsilon,1})v_1 = g + f_{\varepsilon,1}\varepsilon$ and $(I - \varepsilon\, L_{\varepsilon,i+1})v_{i+1} = v_i + f_{\varepsilon,i+1}\varepsilon$, for $i = 0, 1, \ldots, n-1$.

We first check that the operators $I - \varepsilon\, L_{\varepsilon,j+1}$, $j = 0, 1, \ldots, n-1$ satisfy the hypotheses of Lemma 1. These operators are obviously bounded. We have to show that there exists $\lambda > 0$ such that $\langle (I - \varepsilon\, L_{\varepsilon,j+1})v, v\rangle \geq \lambda |v|_V^2$, for all $v \in V$, $j = 0, 1, \ldots, n-1$. Owing to Assumption 5, we have

$$\begin{aligned}\langle (I - \varepsilon\, L_{\varepsilon,j+1})v, v\rangle = \langle Iv - \varepsilon\, L_{\varepsilon,j+1}v, v\rangle &= |v|_H^2 - \varepsilon \langle L_{\varepsilon,j+1}v, v\rangle \\ &\geq |v|_H^2 - \varepsilon K |v|_H^2 + \varepsilon \lambda |v|_V^2.\end{aligned}$$

Then, with ε sufficiently small, $\langle (I - \varepsilon\, L_{\varepsilon,j+1})v, v\rangle \geq \varepsilon \lambda |v|_V^2$, and the hypotheses of Lemma 1 are satisfied.

For v_1 we have that $(I - \varepsilon\, L_{\varepsilon 1})v_1 = g + f_{\varepsilon 1}\varepsilon$. This equation has a unique solution by Lemma 1. Suppose now that equation $(I - \varepsilon\, L_{\varepsilon,i})v_i = v_{i-1} + f_{\varepsilon,i}\varepsilon$ has a unique solution. Then equation $(I - \varepsilon\, L_{\varepsilon,i+1})v_{i+1} = v_i + f_{\varepsilon,i+1}\varepsilon$ has also a unique solution, again by Lemma 1. The result is proved by induction. □

Next, we prove a lemma, and then obtain the discrete version of Gronwall's inequality as a corollary.

Lemma 2. *Let $a_1^n, a_2^n, \ldots, a_n^n$ be a finite sequence of numbers for every integer $n \geq 1$ such that $0 \leq a_j^n \leq c_0 + C \sum_{1 \leq i \leq j-1} a_i^n$, for all $j = 1, 2, \ldots, n$, where C is a positive constant and $c_0 \geq 0$ is some real number. Then $a_j^n \leq (C+1)^{j-1} c_0$, for all $j = 1, 2, \ldots, n$.*

Proof. Let $b_j^n := c_0 + C \sum_{1 \leq i \leq j-1} b_i^n$, $j = 1, 2, \ldots, n$. Then $a_j^n \leq b_j^n$ for all $j \geq 1$. Indeed for $j = 1$ we have that $a_1^n \leq b_1^n = c_0$. Assume now that $a_i^n \leq b_i^n$ for all $i \leq j$. Then

$$b_{j+1}^n = c_0 + C \sum_{1 \leq i \leq j} b_i^n \geq c_0 + C \sum_{1 \leq i \leq j} a_i^n \geq a_{j+1}^n,$$

what proves by induction that $a_j^n \leq b_j^n$ for all $j \geq 1$. It is easy to see that $b_{j+1}^n - b_j^n = C b_j^n$, $j \geq 1$, what gives

$$a_{j+1}^n \leq b_{j+1}^n = (C+1) b_j^n = (C+1)^2\, b_{j-1}^n = \ldots = (C+1)^j\, b_1^n = (C+1)^j\, c_0,$$

and the result is proved. □

Corollary 2. (Discrete Gronwall's inequality). *Let* $a_0^n, a_1^n, \dots, a_n^n$ *be a finite sequence of numbers for every integer* $n \geq 1$ *such that* $0 \leq a_j^n \leq a_0^n + K \sum_{1 \leq i \leq j} a_i^n \varepsilon$ *holds for every* $j = 1, 2, \dots, n$, *with* $\varepsilon := T/n$, *and* K *a positive number such that* $K\varepsilon =: q < 1$, *with* q *a fixed constant. Then* $a_j^n \leq a_0^n e^{K_q T}$, *for all integers* $n \geq 1$ *and* $j = 1, 2, \dots, n$, *where* $K_q := -K \ln(1-q)/q$.

Proof. From the inequality in the hypotheses, as $K\varepsilon < 1$ for $j = 1, 2, \dots, n$ we have that

$$(1 - K\varepsilon) a_j^n \leq a_0^n + K \sum_{1 \leq i \leq j-1} a_i^n \varepsilon \Leftrightarrow a_j^n \leq \frac{a_0^n}{1 - K\varepsilon} + \frac{K\varepsilon}{1 - K\varepsilon} \sum_{1 \leq i \leq j-1} a_i^n.$$

Applying Lemma 2 to the previous inequality with $c_0 = a_0^n/(1 - K\varepsilon)$ and $C = K\varepsilon/(1 - K\varepsilon)$ we obtain

$$a_j^n \leq \Big(\frac{K\varepsilon}{1 - K\varepsilon} + 1\Big)^{j-1} \frac{a_0^n}{1 - K\varepsilon} = \frac{a_0^n}{(1 - K\varepsilon)^j} \leq \frac{a_0^n}{(1 - K\varepsilon)^n}.$$

Noting that

$$(1 - K\varepsilon)^n = \exp(n \ln(1 - K\varepsilon)) = \exp\Big(nK\varepsilon \frac{\ln(1-q)}{q}\Big) = \exp\Big(KT \frac{\ln(1-q)}{q}\Big),$$

the result is proved. □

We are now able to prove that the numerical scheme (5.2) is stable, that is, the solution of the discrete problem remains bounded independently of ε.

Theorem 7. *Let Assumption 5 be satisfied, and denote* $v_{\varepsilon,j}$, *with* $j = 0, 1, \dots, n$, *the unique solution of problem (5.2) in Theorem 6. Assume that constant* K *in Assumption 5 satisfies:* $2K\varepsilon < 1$. *Then there exists a constant* N *independent of* ε *such that*

(1) $\max_{0 \leq j \leq n} |v_{\varepsilon,j}|_H^2 \leq N\big(|g|_H^2 + \sum_{1 \leq j \leq n} |f_{\varepsilon,j}|_{V^*}^2 \varepsilon\big)$;

(2) $\sum_{0 \leq j \leq n} |v_{\varepsilon,j}|_V^2 \varepsilon \leq N\big(|g|_H^2 + \sum_{1 \leq j \leq n} |f_{\varepsilon,j}|_{V^*}^2 \varepsilon\big)$.

Remark 10. Owing to (3) in Assumption 5, the estimates (1) and (2) above can be written $\sup_{n \geq 1} \max_{0 \leq j \leq n} |v_{\varepsilon,j}|_H^2 \leq N$ and $\sup_{n \geq 1} \sum_{0 \leq j \leq n} |v_{\varepsilon,j}|_V^2 \varepsilon \leq N$, respectively.

Proof. (Theorem 7) For $i = 0, 1, \dots, n-1$, we have that

$$|v_{\varepsilon,i+1}|_H^2 - |v_{\varepsilon,i}|_H^2 = 2 \langle v_{\varepsilon,i+1}, v_{\varepsilon,i+1} - v_{\varepsilon,i} \rangle - |v_{\varepsilon,i+1} - v_{\varepsilon,i}|_H^2. \tag{5.3}$$

Summing up both members of equation (5.3) we obtain, for $j = 1, 2, \dots, n$,

$$|v_{\varepsilon,j}|_H^2 = |v_{\varepsilon,0}|_H^2 + \sum_{i=0}^{j-1} 2 \langle v_{\varepsilon,i+1}, v_{\varepsilon,i+1} - v_{\varepsilon,i} \rangle - \sum_{i=0}^{j-1} |v_{\varepsilon,i+1} - v_{\varepsilon,i}|_H^2.$$

Hence

$$\begin{aligned} |v_{\varepsilon,j}|_H^2 &\leq |v_{\varepsilon,0}|_H^2 + \sum_{i=0}^{j-1} 2 \langle v_{\varepsilon,i+1}, v_{\varepsilon,i+1} - v_{\varepsilon,i} \rangle \\ &= |v_{\varepsilon,0}|_H^2 + \sum_{i=0}^{j-1} 2 \langle v_{\varepsilon,i+1}, L_{\varepsilon,i+1} v_{\varepsilon,i+1} \varepsilon + f_{\varepsilon,i+1} \varepsilon \rangle. \end{aligned}$$

As, by Cauchy's inequality,

$$2\langle v_{\varepsilon,i+1}, f_{\varepsilon,i+1}\rangle\varepsilon \le \lambda|v_{\varepsilon,i+1}|^2_V\varepsilon + \frac{1}{\lambda}|f_{\varepsilon,i+1}|^2_{V^*}\varepsilon,$$

with $\lambda > 0$, owing to (1) in Assumption 5 we obtain

$$|v_{\varepsilon,j}|^2_H \le |v_{\varepsilon,0}|^2_H + 2K\sum_{i=0}^{j-1}|v_{\varepsilon,i+1}|^2_H\varepsilon - \lambda\sum_{i=0}^{j-1}|v_{\varepsilon,i+1}|^2_V\varepsilon + \frac{1}{\lambda}\sum_{i=0}^{j-1}|f_{\varepsilon,i+1}|^2_{V^*}\varepsilon$$

and then

$$|v_{\varepsilon,j}|^2_H + \lambda\sum_{i=1}^{j}|v_{\varepsilon,i}|^2_V\varepsilon \le |v_{\varepsilon,0}|^2_H + 2K\sum_{i=1}^{j}|v_{\varepsilon,i}|^2_H\varepsilon + \frac{1}{\lambda}\sum_{i=1}^{n}|f_{\varepsilon,i}|^2_{V^*}\varepsilon. \tag{5.4}$$

In particular

$$|v_{\varepsilon,j}|^2_H \le |v_{\varepsilon,0}|^2_H + 2K\sum_{i=1}^{j}|v_{\varepsilon,i}|^2_H\varepsilon + \frac{1}{\lambda}\sum_{i=1}^{n}|f_{\varepsilon,i}|^2_{V^*}\varepsilon, \tag{5.5}$$

and, using Corollary 2,

$$|v_{\varepsilon,j}|^2_H \le \Big(|v_{\varepsilon,0}|^2_H + \frac{1}{\lambda}\sum_{i=1}^{n}|f_{\varepsilon,i}|^2_{V^*}\varepsilon\Big)e^{2K_qT}, \tag{5.6}$$

where K_q is the constant defined in the Corollary. Estimate (1) follows.

From (5.4), (5.5) and (5.6) we obtain

$$|v_{\varepsilon,j}|^2_H + \lambda\sum_{i=1}^{j}|v_{\varepsilon,i}|^2_V\varepsilon \le \Big(|v_{\varepsilon,0}|^2_H + \frac{1}{\lambda}\sum_{i=1}^{n}|f_{\varepsilon,i}|^2_{V^*}\varepsilon\Big)e^{2K_qT}$$

and

$$\sum_{i=1}^{j}|v_{\varepsilon,i}|^2_V\varepsilon \le \Big(|v_{\varepsilon,0}|^2_H + \frac{1}{\lambda}\sum_{i=1}^{n}|f_{\varepsilon,i}|^2_{V^*}\varepsilon\Big)\frac{1}{\lambda}\,e^{2K_qT}.$$

Estimate (2) follows. □

We will now study the convergence properties of the numerical scheme we have constructed.

We impose stronger smoothness over the solution $u = u(t)$ of problem (2.1):

Assumption 6. Let u be the solution of problem (2.1) in Theorem 1. There exist a fixed number $\delta \in (0, 1]$ and a constant C such that

$$\frac{1}{\varepsilon}\int_{t_i}^{t_{i+1}}|u(t_{i+1}) - u(s)|_V ds \le C\varepsilon^\delta,$$

for all $i = 0, 1, \ldots, n-1$.

Remark 11. Assume that u satisfies the following condition: "There exists a fixed number $\delta \in (0, 1]$ and a constant C such that $|u(t) - u(s)|_V \le C|t-s|^\delta$, for all $s, t \in [0, T]$". Then Assumption 6 obviously holds.

Assuming this stronger regularity of the solution u of (2.1), we can prove the convergence of the solution of problem (5.2) to the solution of problem (2.1), and determine the convergence rate. The accuracy we obtain is of order δ.

Theorem 8. *Let Assumptions 1 and 5 be satisfied. Denote $u(t)$ the unique solution of (2.1) in Theorem 1, and $v_{\varepsilon,j}$, $j = 0, 1, \dots, n$, the unique solution of (5.2) in Theorem 6. Let Assumption 6 be satisfied, and assume that constant K in Assumptions 1 and 5 satisfies: $2K\varepsilon < 1$. Then there exists a constant N independent of ε such that*

(1) $\max_{0\le j\le n} |v_{\varepsilon,j} - u(t_j)|_H^2$

$$\le N\Big(\varepsilon^{2\delta} + \sum_{j=1}^{n} \frac{1}{\varepsilon}\Big|L_{\varepsilon,j}u(t_j)\varepsilon - \int_{t_{j-1}}^{t_j} L(s)u(t_j)ds\Big|_{V^*}^2 + \sum_{j=1}^{n}\frac{1}{\varepsilon}\Big|f_{\varepsilon,j}\varepsilon - \int_{t_{j-1}}^{t_j} f(s)ds\Big|_{V^*}^2\Big);$$

(2) $\sum_{j=0}^{n} |v_{\varepsilon,j} - u(t_j)|_V^2\varepsilon$

$$\le N\Big(\varepsilon^{2\delta} + \sum_{j=1}^{n} \frac{1}{\varepsilon}\Big|L_{\varepsilon,j}u(t_j)\varepsilon - \int_{t_{j-1}}^{t_j} L(s)u(t_j)ds\Big|_{V^*}^2 + \sum_{j=1}^{n}\frac{1}{\varepsilon}\Big|f_{\varepsilon,j}\varepsilon - \int_{t_{j-1}}^{t_j} f(s)ds\Big|_{V^*}^2\Big).$$

Proof. Define $w(t_i) := v_{\varepsilon,i} - u(t_i)$, $i = 0, 1, \dots, n$. For $i = 0, 1, \dots, n-1$,

$$\begin{aligned} w(t_{i+1}) - w(t_i) &= L_{\varepsilon,i+1}w(t_{i+1})\varepsilon + f_{\varepsilon,i+1}\varepsilon - u(t_{i+1}) + u(t_i) + L_{\varepsilon,i+1}u(t_{i+1})\varepsilon \\ &= L_{\varepsilon,i+1}w(t_{i+1})\varepsilon + \varphi(t_{i+1}), \end{aligned}$$

where $\varphi(t_{i+1}) := f_{\varepsilon,i+1}\varepsilon - u(t_{i+1}) + u(t_i) + L_{\varepsilon,i+1}u(t_{i+1})\varepsilon$.

We obtain, owing to (1) in Assumption 5,

$$\begin{aligned} |w(t_{i+1})|_H^2 - |w(t_i)|_H^2 &= 2\langle w(t_{i+1}), w(t_{i+1}) - w(t_i)\rangle - |w(t_{i+1}) - w(t_i)|_H^2 \\ &\le 2\langle w(t_{i+1}), L_{\varepsilon,i+1}w(t_{i+1})\rangle\varepsilon + 2\langle w(t_{i+1}), \varphi(t_{i+1})\rangle \\ &\le -2\lambda|w(t_{i+1})|_V^2\varepsilon + 2K|w(t_{i+1})|_H^2\varepsilon \\ &\quad + 2|\langle w(t_{i+1}), \varphi(t_{i+1})\rangle|. \end{aligned} \tag{5.7}$$

Noting that $\varphi(t_{i+1})$ can be written

$$\varphi(t_{i+1}) = \int_{t_i}^{t_{i+1}} L(s)(u(t_{i+1}) - u(s))ds + \varphi_1(t_{i+1}) + \varphi_2(t_{i+1}),$$

where

$$\varphi_1(t_{i+1}) := L_{\varepsilon,i+1}u(t_{i+1})\varepsilon - \int_{t_i}^{t_{i+1}} L(s)u(t_{i+1})ds \quad \text{and} \quad \varphi_2(t_{i+1}) := f_{\varepsilon,i+1}\varepsilon - \int_{t_i}^{t_{i+1}} f(s)ds,$$

for the last term in (5.7) we have the estimate

$$\begin{aligned} 2|\langle w(t_{i+1}), \varphi(t_{i+1})\rangle| \le\ & 2\Big|\Big\langle w(t_{i+1}), \int_{t_i}^{t_{i+1}} L(s)(u(t_{i+1}) - u(s))ds\Big\rangle\Big| \\ & + 2|\langle w(t_{i+1}), \varphi_1(t_{i+1})\rangle| + 2|\langle w(t_{i+1}), \varphi_2(t_{i+1})\rangle|. \end{aligned} \tag{5.8}$$

Now, we estimate separately each one of the three terms in (5.8).

For the first term, owing to (2) in Assumption 1 and using Cauchy-Schwarz and Cauchy's inequalities, we obtain

$$
\begin{aligned}
&2\big|\langle w(t_{i+1}), \int_{t_i}^{t_{i+1}} L(s)(u(t_{i+1}) - u(s))ds\rangle\big| \\
&\leq 2\int_{t_i}^{t_{i+1}} |\langle w(t_{i+1}), L(s)(u(t_{i+1}) - u(s))\rangle| ds \\
&\leq 2M|w(t_{i+1})|_V \int_{t_i}^{t_{i+1}} |u(t_{i+1}) - u(s)|_V ds \\
&\leq \frac{\lambda}{3}\,|w(t_{i+1})|_V^2\varepsilon + \frac{3M^2}{\lambda\varepsilon}\Big(\int_{t_i}^{t_{i+1}} |u(t_{i+1}) - u(s)|_V ds\Big)^2,
\end{aligned}
\tag{5.9}
$$

with $\lambda > 0$.

For the two remaining terms, we have the estimates

$$
2|\langle w(t_{i+1}), \varphi_1(t_{i+1})\rangle| \leq \frac{\lambda}{3}|w(t_{i+1})|_V^2\varepsilon + \frac{3}{\lambda\varepsilon}|\varphi_1(t_{i+1})|_{V^*}^2,
\tag{5.10}
$$

and

$$
2|\langle w(t_{i+1}), \varphi_2(t_{i+1})\rangle| \leq \frac{\lambda}{3}|w(t_{i+1})|_V^2\varepsilon + \frac{3}{\lambda\varepsilon}|\varphi_2(t_{i+1})|_{V^*}^2,
\tag{5.11}
$$

with $\lambda > 0$, using Cauchy's inequality.

Therefore, from (5.8), (5.9), (5.10) and (5.11), we get

$$
\begin{aligned}
2|\langle w(t_{i+1}), \varphi(t_{i+1})\rangle| \leq \lambda|w(t_{i+1})|_V^2\varepsilon &+ \frac{3M^2}{\lambda\varepsilon}\Big(\int_{t_i}^{t_{i+1}}|u(t_{i+1}) - u(s)|_V ds\Big)^2 \\
&+ \frac{3}{\lambda\varepsilon}|\varphi_1(t_{i+1})|_{V^*}^2 + \frac{3}{\lambda\varepsilon}|\varphi_2(t_{i+1})|_{V^*}^2.
\end{aligned}
\tag{5.12}
$$

Putting estimates (5.7) and (5.12) together and summing up, owing to Assumption 6 we obtain, for $j = 1, 2, \dots, n$,

$$
\begin{aligned}
|w(t_j)|_H^2 + \lambda\sum_{i=0}^{j-1} |w(t_{i+1})|_V^2\varepsilon \leq 2K\sum_{i=0}^{j-1} |w(t_{i+1})|_H^2\varepsilon &+ \frac{3M^2}{\lambda}\sum_{i=0}^{j-1}\varepsilon^{2\delta+1} \\
&+ \frac{3}{\lambda\varepsilon}\sum_{i=0}^{j-1} |\varphi_1(t_{i+1})|_{V^*}^2 + \frac{3}{\lambda\varepsilon}\sum_{i=0}^{j-1} |\varphi_2(t_{i+1})|_{V^*}^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|w(t_j)|_H^2 + \lambda\sum_{i=1}^{j} |w(t_i)|_V^2\varepsilon \leq 2K\sum_{i=1}^{j} |w(t_i)|_H^2\varepsilon &+ N\varepsilon^{2\delta} \\
&+ N\sum_{i=1}^{n}\frac{1}{\varepsilon}\big|L_{\varepsilon,i}u(t_i)\varepsilon - \int_{t_{i-1}}^{t_i} L(s)u(t_i)ds\big|_{V^*}^2 \\
&+ N\sum_{i=1}^{n}\frac{1}{\varepsilon}\big|f_{\varepsilon,i}\varepsilon - \int_{t_{i-1}}^{t_i} f(s)ds\big|_{V^*}^2,
\end{aligned}
$$

with N a constant. Following the same steps as in the proof of Theorem 7, estimates (1) and (2) follow. □

Next result is an immediate consequence of Theorem 8.

Corollary 3. *Let the hypotheses of Theorem 8 be satisfied, and denote $u(t)$ the unique solution of (2.1) in Theorem 1, and $v_{\varepsilon,j}$, $j = 0, 1, \ldots, n$, the unique solution of (5.2) in Theorem 6. If there exists a constant N independent of ε such that*

$$\big|L_{\varepsilon,j}u(t_j) - \frac{1}{\varepsilon}\int_{t_{j-1}}^{t_j} L(s)u(t_j)ds\big|^2_{V^*} + \big|f_{\varepsilon,j} - \frac{1}{\varepsilon}\int_{t_{j-1}}^{t_j} f(s)ds\big|^2_{V^*} \le N\varepsilon^{2\delta},$$

for $j = 1, 2, \ldots, n$, then

$$\max_{0\le j\le n} |v_{\varepsilon,j} - u(t_j)|^2_H \le N\varepsilon^{2\delta} \quad \text{and} \quad \sum_{0\le j\le n} |v_{\varepsilon,j} - u(t_j)|^2_V \varepsilon \le N\varepsilon^{2\delta}.$$

Remark 12. In the special case where L and f in problem (2.1) are approximated by the average operators

$$\bar{L}_\varepsilon(t_{j+1})z := \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} L(s)zds \quad \text{and} \quad \bar{f}_\varepsilon(t_{j+1}) := \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} f(s)ds$$

for all $z \in V$, $j = 0, 1, \ldots, n-1$, respectively, the estimates in Corollary 3 are obtained without the need of any further assumption. Additionally, in this special case, under Assumption 1 Assumption 5 is satisfied.

6. Numerical approximation in time: explicit scheme

We now approach the time-discretization with the use of a finite-difference explicit scheme. As in the previous Section, we begin by setting a suitable discrete framework, and then investigate the stability and convergence properties of the numerical scheme.

Observe that, when using the explicit scheme, a previous discretization in space has to be assumed. We mirror this fact in our approximation study of the general linear evolution equation by considering the version of problem (2.1)

$$L_h(t)u(t) - \frac{\partial u(t)}{\partial t} + f_h(t) = 0, \quad u(0) = g_h, \tag{6.1}$$

in the spaces V_h and H_h, "space-discrete versions" of V and H, and with $L_h(t)$, $f_h(t)$ and g_h "space-discrete versions" of $L(t)$, $f(t)$ and g, respectively.

Let the time-grid T_n as defined in (5.1). For all $z \in V_h$, consider the forward discrete difference in time

$$\Delta^+ z(t_j) = \varepsilon^{-1}(z(t_{j+1}) - z(t_j)), \quad j = 0, 1, \ldots, n-1.$$

Let $L_{h\varepsilon}$, $f_{h\varepsilon}$ be some time-discrete versions of L_h and f_h, respectively, and denote, for all $z \in V_h$,

$$L_{h\varepsilon,j+1}z = L_{h\varepsilon}(t_{j+1})z, \quad f_{h\varepsilon,j+1} = f_{h\varepsilon}(t_{j+1}),$$

with $j = 0, 1, \ldots, n-1$.

For each $n \ge 1$ fixed, we consider the time-discrete version of (6.1),

$$\Delta^+ v_i = L_{h\varepsilon,i}\, v_i + f_{h\varepsilon,i} \ \text{ for } \ i = 0, 1, \ldots, n-1, \qquad v_0 = g_h, \tag{6.2}$$

with $v_j = v(t_j)$, $j = 0, 1, \ldots, n$, in V_h.

Problem (6.2) can be solved uniquely by recursion

$$v_j = g_h + \sum_{i=0}^{j-1} L_{h\varepsilon,i}\, v_i\varepsilon + \sum_{i=0}^{j-1} f_{h\varepsilon,i}\varepsilon \ \text{ for } \ j = 1, \ldots, n, \qquad v_0 = g_h.$$

We make some assumptions.

Assumption 7. We assume

(1) $\langle L_{h\varepsilon,j}v, v\rangle_h + \lambda|v|^2_{V_h} \le K|v|^2_{H_h}, \quad \forall v \in V_h,\ \ j = 0, 1, \dots, n-1$
(2) $|L_{h\varepsilon,j}v|_{V_h^*} \le M|v|_{V_h}, \quad \forall v \in V_h,\ \ j = 0, 1, \dots, n-1$
(3) $\sum_{j=0}^{n-1} |f_{h\varepsilon,j}|^2_{V_h^*}\varepsilon \le N$ and $|g_h|_{H_h} \le N$,

where λ, K, M and N are the constants in Assumption 1.

Remark 13. We refer to Remark 9 to note that under Assumption 1 $g_h \in H_h$ and $|g_h|_{H_h} \le N$.

The following version of the discrete Gronwall's inequality is an immediate consequence of Corollary 2:

Lemma 3. *Let $a_0^n, a_1^n, \dots, a_n^n$ be a finite sequence of numbers for every integer $n \ge 1$ such that $0 \le a_j^n \le a_0^n + K\sum_{0\le i\le j-1} a_i^n\varepsilon$, holds for every $j = 0, 1, \dots, n$, with $\varepsilon := T/n$ and K a positive number such that $K\varepsilon =: q < 1$, with q a fixed constant. Then $a_j^n \le a_0^n e^{K_qT}$, for all integers $n \ge 1$ and $j = 0, 1, \dots, n$, where $K_q := -K\ln(1-q)/q$.*

In order to obtain stability for the numerical scheme (6.2) we need to make the usual additional assumption, establishing a relation between the time and space steps. We note that, for the case of the implicit scheme, there was no such need: the stability of the implicit scheme was met unconditionally.

Assumption 8. There exists a constant C_h, dependent of the space-step h, such that $|w|_{H_h} \le C_h|w|_{V_h^*}$ for all $w \in V_h$.

We now investigate the numerical scheme's stability.

Theorem 9. *Let Assumptions 7 and 8 be satisfied, and λ, K, M, C_h the constants defined in the Assumptions. Denote by $v_{h\varepsilon,j}$, with $j = 0, 1, \dots, n$, the unique solution of problem (6.2). Assume that constant K satisfies: $2K\varepsilon < 1$. If there exists a number p such that $M^2C_h^2\varepsilon \le p < \lambda$ then there exists a constant N, independent of ε and h, such that*

(1) $\max_{0\le j\le n} |v_{h\varepsilon,j}|^2_{H_h} \le N\big(|g_h|^2_{H_h} + \sum_{0\le j\le n-1} |f_{h\varepsilon,j}|^2_{V_h^*}\varepsilon\big)$;
(2) $\sum_{0\le j\le n} |v_{h\varepsilon,j}|^2_{V_h}\varepsilon \le N\big(|g_h|^2_{H_h} + \sum_{0\le j\le n-1} |f_{h\varepsilon,j}|^2_{V_h^*}\varepsilon\big)$.

Remark 14. Remark 10 applies to the above theorem with the obvious adaptations.

Proof. (*Theorem 9*) For $i = 0, 1, \dots, n-1$, we have

$$|v_{h\varepsilon,i+1}|^2_{H_h} - |v_{h\varepsilon,i}|^2_{H_h} = 2\,\langle v_{h\varepsilon,i}, v_{h\varepsilon,i+1} - v_{h\varepsilon,i}\rangle_h + |v_{h\varepsilon,i+1} - v_{h\varepsilon,i}|^2_{H_h}. \tag{6.3}$$

Summing up both members of equation (6.3), for $j = 1, 2, \dots, n$,

$$\begin{aligned} |v_{h\varepsilon,j}|^2_{H_h} &= |v_{h\varepsilon,0}|^2_{H_h} + \sum_{i=0}^{j-1} 2\langle v_{h\varepsilon,i}, v_{h\varepsilon,i+1} - v_{h\varepsilon,i}\rangle_h + \sum_{i=0}^{j-1} |v_{h\varepsilon,i+1} - v_{h\varepsilon,i}|^2_{H_h} \\ &= |v_{h\varepsilon,0}|^2_{H_h} + \sum_{i=0}^{j-1} 2\langle v_{h\varepsilon,i}, L_{h\varepsilon,i}v_{h\varepsilon,i}\rangle_h\,\varepsilon + \sum_{i=0}^{j-1} 2\langle v_{h\varepsilon,i}, f_{h\varepsilon,i}\rangle_h\,\varepsilon \\ &\quad + \sum_{i=0}^{j-1} |L_{h\varepsilon,i}v_{h\varepsilon,i} + f_{h\varepsilon,i}|^2_{H_h}\varepsilon^2. \end{aligned} \tag{6.4}$$

Owing to (1) in Assumption 7 and using Cauchy's inequality, from (6.4) we obtain the estimate

$$
\begin{aligned}
(6.5)\qquad |v_{h\varepsilon,j}|_{H_h}^2 \leq & |v_{h\varepsilon,0}|_{H_h}^2 + 2K\sum_{i=0}^{j-1}|v_{h\varepsilon,i}|_{H_h}^2\varepsilon - 2\lambda\sum_{i=0}^{j-1}|v_{h\varepsilon,i}|_{V_h}^2\varepsilon + \lambda\sum_{i=0}^{j-1}|v_{h\varepsilon,i}|_{V_h}^2\varepsilon \\
& + \frac{1}{\lambda}\sum_{i=0}^{j-1}|f_{h\varepsilon,i}|_{V_h^*}^2\varepsilon + \sum_{i=0}^{j-1}|L_{h\varepsilon,i}v_{h\varepsilon,i} + f_{h\varepsilon,i}|_{H_h}^2\varepsilon^2,
\end{aligned}
$$

with $\lambda > 0$.

For the last term in the above estimate (6.5), we have

$$
\begin{aligned}
& \sum_{i=0}^{j-1}|L_{h\varepsilon,i}v_{h\varepsilon,i} + f_{h\varepsilon,i}|_{H_h}^2\varepsilon^2 \\
& = \sum_{i=0}^{j-1}|L_{h\varepsilon,i}v_{h\varepsilon,i}|_{H_h}^2\varepsilon^2 + \sum_{i=0}^{j-1}|f_{h\varepsilon,i}|_{H_h}^2\varepsilon^2 + 2\sum_{i=0}^{j-1}\langle f_{h\varepsilon,i}, L_{h\varepsilon,i}v_{h\varepsilon,i}\rangle_h\,\varepsilon^2 \\
& \leq \sum_{i=0}^{j-1}|L_{h\varepsilon,i}v_{h\varepsilon,i}|_{H_h}^2\varepsilon^2 + \sum_{i=0}^{j-1}|f_{h\varepsilon,i}|_{H_h}^2\varepsilon^2 + \frac{1}{\mu}\sum_{i=0}^{j-1}|f_{h\varepsilon,i}|_{H_h}^2\varepsilon^2 + \mu\sum_{i=0}^{j-1}|L_{h\varepsilon,i}v_{h\varepsilon,i}|_{H_h}^2\varepsilon^2.
\end{aligned}
$$

with $\mu > 0$, using Cauchy's inequality.

As, owing to (2) in Assumption 7 and to Assumption 8,

$$
\sum_{i=0}^{j-1}|L_{h\varepsilon,i}v_{h\varepsilon,i}|_{H_h}^2\varepsilon^2 \leq C_h^2\varepsilon\sum_{i=0}^{j-1}|L_{h\varepsilon,i}v_{h\varepsilon,i}|_{V_h^*}^2\varepsilon \leq M^2C_h^2\varepsilon\sum_{i=0}^{j-1}|v_{h\varepsilon,i}|_{V_h}^2\varepsilon,
$$

and

$$
\sum_{i=0}^{j-1}|f_{h\varepsilon,i}|_{H_h}^2\varepsilon^2 \leq C_h^2\varepsilon\sum_{i=0}^{j-1}|f_{h\varepsilon,i}|_{V_h^*}^2\varepsilon,
$$

we then obtain

$$
\begin{aligned}
(6.6)\qquad & \sum_{i=0}^{j-1}|L_{h\varepsilon,i}v_{h\varepsilon,i} + f_{h\varepsilon,i}|_{H_h}\varepsilon^2 \\
& \leq\ (1+\mu)M^2C_h^2\varepsilon\sum_{i=0}^{j-1}|v_{h\varepsilon,i}|_{V_h}^2\varepsilon + \Big(1+\frac{1}{\mu}\Big)C_h^2\varepsilon\sum_{i=0}^{j-1}|f_{h\varepsilon,i}|_{V_h^*}^2\varepsilon.
\end{aligned}
$$

Putting together estimates (6.5) and (6.6), we get

$$
\begin{aligned}
(6.7)\qquad |v_{h\varepsilon,j}|_{H_h}^2 \leq & |v_{h\varepsilon,0}|_{H_h}^2 + 2K\sum_{i=0}^{j-1}|v_{h\varepsilon,i}|_{H_h}^2\varepsilon + ((1+\mu)M^2C_h^2\varepsilon - \lambda)\sum_{i=0}^{j-1}|v_{h\varepsilon,i}|_{V_h}^2\varepsilon \\
& + \Big(\frac{1}{\lambda} + \Big(1+\frac{1}{\mu}\Big)C_h^2\varepsilon\Big)\sum_{i=0}^{j-1}|f_{h\varepsilon,i}|_{V_h^*}^2\varepsilon.
\end{aligned}
$$

Now, if there is a constant p such that

$$
M^2C_h^2\varepsilon \leq p < \lambda,
$$

implying that, for μ sufficiently small,

$$
(1+\mu)M^2C_h^2\varepsilon - \lambda \leq (1+\mu)p - \lambda < 0,
$$

then from (6.7) we obtain the estimate

$$
\begin{aligned}
&|v_{h\varepsilon,j}|^2_{H_h} + (\lambda - (1+\mu)p) \sum_{i=0}^{j-1} |v_{h\varepsilon,i}|^2_{V_h} \varepsilon \\
&\le |v_{h\varepsilon,0}|^2_{H_h} + 2K \sum_{i=0}^{j-1} |v_{h\varepsilon,i}|^2_{H_h} \varepsilon + L \sum_{i=0}^{n-1} |f_{h\varepsilon,i}|^2_{V_h^*} \varepsilon,
\end{aligned}
\tag{6.8}
$$

where $L := (\mu M^2 + \lambda(1+\mu)p)/\lambda\mu M^2$.

In particular,

$$
|v_{h\varepsilon,j}|^2_{H_h} \le |v_{h\varepsilon,0}|^2_{H_h} + 2K \sum_{i=0}^{j-1} |v_{h\varepsilon,i}|^2_{H_h} \varepsilon + L \sum_{i=0}^{n-1} |f_{h\varepsilon,i}|^2_{V_h^*} \varepsilon,
\tag{6.9}
$$

and, using Lemma 3,

$$
|v_{h\varepsilon,j}|^2_{H_h} \le \Big(|v_{h\varepsilon,0}|^2_{H_h} + L \sum_{i=0}^{n-1} |f_{h\varepsilon,i}|^2_{V_h^*} \varepsilon \Big) e^{2K_q T},
\tag{6.10}
$$

where K_q is the constant defined in Lemma 3. (1) follows.

From (6.8), (6.9) and (6.10) we obtain

$$
|v_{h\varepsilon,j}|^2_{H_h} + (\lambda - (1+\mu)p) \sum_{i=0}^{j-1} |v_{h\varepsilon,i}|^2_{V_h} \varepsilon \le \Big(|v_{h\varepsilon,0}|^2_{H_h} + L \sum_{i=0}^{n-1} |f_{h\varepsilon,i}|^2_{V_h^*} \varepsilon \Big) e^{2K_q T},
$$

and (2) follows. □

Finally, we prove the convergence of the numerical scheme, and determine the convergence rate. The accuracy obtained is of order δ, with δ given by Assumption 6.

Theorem 10. *Let Assumptions 1, 7 and 8 be satisfied, and λ, K, M, C_h the constants defined in the Assumptions. Denote by $u_h(t)$ the unique solution of problem (6.1) in Theorem 1, and by $v_{h\varepsilon,j}$, with $j = 0, 1, \dots, n$, the unique solution of problem (6.2). Assume that constant K is such that $2K\varepsilon < 1$, and that Assumption 6 is satisfied. If there exists a number p such that $M^2 C_h^2 \varepsilon \le p < \lambda$ then there exists a constant N, independent of ε and h, such that*

(1) $\max_{0 \le j \le n} |v_{h\varepsilon,j} - u_h(t_j)|^2_{H_h}$

$$
\le N\Big(\varepsilon^{2\delta} + \sum_{j=0}^{n-1} \frac{1}{\varepsilon} \Big| L_{h\varepsilon,j} u_h(t_j)\varepsilon - \int_{t_j}^{t_{j+1}} L_h(s) u_h(t_j) ds \Big|^2_{V_h^*} + \sum_{j=0}^{n-1} \frac{1}{\varepsilon} \Big| f_{h\varepsilon,j}\varepsilon - \int_{t_j}^{t_{j+1}} f_h(s) ds \Big|^2_{V_h^*} \Big);
$$

(2) $\sum_{j=0}^{n} |v_{h\varepsilon,j} - u_h(t_j)|^2_{V_h} \varepsilon$

$$
\le N\Big(\varepsilon^{2\delta} + \sum_{j=0}^{n-1} \frac{1}{\varepsilon} \Big| L_{h\varepsilon,j} u_h(t_j)\varepsilon - \int_{t_j}^{t_{j+1}} L_h(s) u_h(t_j) ds \Big|^2_{V_h^*} + \sum_{j=0}^{n-1} \frac{1}{\varepsilon} \Big| f_{h\varepsilon,j}\varepsilon - \int_{t_j}^{t_{j+1}} f_h(s) ds \Big|^2_{V_h^*} \Big).
$$

Proof. Define $w(t_i) := v_{h\varepsilon,i} - u_h(t_i)$, $i = 0, 1, \dots, n$. For $i = 0, 1, \dots, n-1$

$$
\begin{aligned}
w(t_{i+1}) - w(t_i) &= L_{h\varepsilon,i} w(t_i)\varepsilon + f_{h\varepsilon,i}\varepsilon - u_h(t_{i+1}) + u_h(t_i) + L_{h\varepsilon,i} u_h(t_i)\varepsilon \\
&= L_{h\varepsilon,i} w(t_i)\varepsilon + \varphi(t_i),
\end{aligned}
$$

denoting $\varphi(t_i) := f_{h\varepsilon,i}\varepsilon - u_h(t_{i+1}) + u_h(t_i) + L_{h\varepsilon,i}\, u_h(t_i)\varepsilon$.

We have that

$$\begin{aligned} &|w(t_{i+1})|^2_{H_h} - |w(t_i)|^2_{H_h} \\ &= 2\langle w(t_i), w(t_{i+1}) - w(t_i)\rangle_h + |w(t_{i+1}) - w(t_i)|^2_{H_h} \\ &= 2\langle w(t_i), L_{h\varepsilon,i} w(t_i)\rangle_h \varepsilon + 2\, |\langle w(t_i), \varphi(t_i)\rangle_h| + |L_{h\varepsilon,i} w(t_i)\varepsilon + \varphi(t_i)|^2_{H_h}. \end{aligned} \tag{6.11}$$

We want to estimate each one of the three terms in (6.11). For the first term in (6.11), owing to (1) in Assumption 7, we obtain

$$2\, \langle w(t_i), L_{h\varepsilon,i} w(t_i)\rangle_h \varepsilon \le -2\lambda |w(t_i)|^2_{V_h}\varepsilon + 2K|w(t_i)|^2_{H_h}\varepsilon. \tag{6.12}$$

Noting that $\varphi(t_i)$ can be written

$$\varphi(t_i) = \int_{t_i}^{t_{i+1}} L_h(s)(u_h(t_i) - u_h(s))ds + \varphi_1(t_i) + \varphi_2(t_i),$$

where

$$\varphi_1(t_i) := L_{h\varepsilon,i} u_h(t_i)\varepsilon - \int_{t_i}^{t_{i+1}} L_h(s) u_h(t_i) ds \quad \text{and} \quad \varphi_2(t_i) := f_{h\varepsilon,i}\varepsilon - \int_{t_i}^{t_{i+1}} f_h(s)ds,$$

for the second term in (6.11) we have

$$\begin{aligned} 2\, |\langle w(t_i), \varphi(t_i)\rangle_h| \le 2\Big|\Big\langle w(t_i), \int_{t_i}^{t_{i+1}} L_h(s)(u_h(t_i) - u_h(s))ds\Big\rangle_h\Big| \\ + 2|\langle w(t_i), \varphi_1(t_i)\rangle_h| + 2|\langle w(t_i), \varphi_2(t_i)\rangle_h| \end{aligned} \tag{6.13}$$

and, following the same steps as in the proof of Theorem 8, we obtain the estimate

$$\begin{aligned} 2|\langle w(t_i), \varphi(t_i)\rangle_h| \le \lambda |w(t_i)|^2_{V_h}\varepsilon + \frac{3M^2}{\lambda\varepsilon}\Big(\int_{t_i}^{t_{i+1}} |u_h(t_i) - u_h(s)|_{V_h} ds\Big)^2 \\ + \frac{3}{\lambda\varepsilon}|\varphi_1(t_i)|^2_{V_h^*} + \frac{3}{\lambda\varepsilon}|\varphi_2(t_i)|^2_{V_h^*}. \end{aligned} \tag{6.14}$$

The last term in (6.11) can be written

$$|L_{h\varepsilon,i} w(t_i)\varepsilon + \varphi(t_i)|^2_{H_h} = |L_{h\varepsilon,i} w(t_i)|^2_{H_h}\varepsilon^2 + |\varphi(t_i)|^2_{H_h} + 2\langle L_{h\varepsilon,i} w(t_i), \varphi(t_i)\rangle_h \varepsilon. \tag{6.15}$$

We estimate the terms in (6.15). For the first term, owing to (2) in Assumption 7 and to Assumption 8,

$$|L_{h\varepsilon,i} w(t_i)|^2_{H_h}\varepsilon^2 \le C_h^2 |L_{h\varepsilon,i} w(t_i)|^2_{V_h^*}\varepsilon^2 \le M^2 C_h^2 \varepsilon |w(t_i)|^2_{V_h}\varepsilon. \tag{6.16}$$

With respect to the second term, first note that it can be written

$$\begin{aligned} |\varphi(t_i)|^2_{H_h} =& \Big|\int_{t_i}^{t_{i+1}} L_h(s)(u_h(t_i) - u_h(s))ds\Big|^2_{H_h} + |\varphi_1(t_i)|^2_{H_h} + |\varphi_2(t_i)|^2_{H_h} \\ &+ 2\Big\langle \int_{t_i}^{t_{i+1}} L_h(s)(u_h(t_i) - u_h(s))ds, \varphi_1(t_i)\Big\rangle_h \\ &+ 2\Big\langle \int_{t_i}^{t_{i+1}} L_h(s)(u_h(t_i) - u_h(s))ds, \varphi_2(t_i)\Big\rangle_h + 2\langle \varphi_1(t_i), \varphi_2(t_i)\rangle_h. \end{aligned}$$

Using Cauchy-Schwarz and Cauchy's inequalities, and owing to (2) in Assumption 1 and to Assumption 8 we get

$$
\begin{aligned}
|\varphi(t_i)|^2_{H_h} \le{}& M^2C_h^2\Big(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\Big)^2 + C_h^2|\varphi_1(t_i)|^2_{V_h^*} + C_h^2|\varphi_2(t_i)|^2_{V_h^*}\\
&+\mu M^2C_h^2\Big(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\Big)^2 + \frac{1}{\mu}C_h^2|\varphi_1(t_i)|^2_{V_h^*}\\
&+\frac{1}{\mu}M^2C_h^2\Big(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\Big)^2 + \mu C_h^2|\varphi_2(t_i)|^2_{V_h^*}\\
&+\mu C_h^2|\varphi_1(t_i)|^2_{V_h^*} + \frac{1}{\mu}C_h^2|\varphi_2(t_i)|^2_{V_h^*}\\
={}&\Big(1+\mu+\frac{1}{\mu}\Big)M^2C_h^2\Big(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\Big)^2\\
&+\Big(1+\mu+\frac{1}{\mu}\Big)C_h^2|\varphi_1(t_i)|^2_{V_h^*} + \Big(1+\mu+\frac{1}{\mu}\Big)C_h^2|\varphi_2(t_i)|^2_{V_h^*},
\end{aligned}
\tag{6.17}
$$

with $\mu > 0$.

For the last term,

$$
\begin{aligned}
&2\langle L_{h\varepsilon,i}w(t_i),\varphi(t_i)\rangle_h\varepsilon\\
&\le 2\Big|\Big\langle L_{h\varepsilon,i}w(t_i)\varepsilon, \int_{t_i}^{t_{i+1}}L_h(s)(u_h(t_i)-u_h(s))ds\Big\rangle_h\Big|\\
&\quad + 2|\langle L_{h\varepsilon,i}w(t_i)\varepsilon,\varphi_1(t_i)\rangle_h| + 2|\langle L_{h\varepsilon,i}w(t_i)\varepsilon,\varphi_2(t_i)\rangle_h|\\
&\le \frac{\nu}{3}M^2C_h^2\varepsilon|w(t_i)|^2_{V_h}\varepsilon + \frac{3}{\nu}M^2C_h^2\Big(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\Big)^2 + \frac{\nu}{3}M^2C_h^2\varepsilon|w(t_i)|^2_{V_h}\varepsilon\\
&\quad + \frac{3}{\nu}C_h^2|\varphi_1(t_i)|^2_{V_h^*} + \frac{\nu}{3}M^2C_h^2\varepsilon|w(t_i)|^2_{V_h}\varepsilon + \frac{3}{\nu}C_h^2|\varphi_2(t_i)|^2_{V_h^*}\\
&= \nu M^2C_h^2\varepsilon|w(t_i)|^2_{V_h}\varepsilon + \frac{3}{\nu}M^2C_h^2\Big(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\Big)^2\\
&\quad + \frac{3}{\nu}C_h^2|\varphi_1(t_i)|^2_{V_h^*} + \frac{3}{\nu}C_h^2|\varphi_2(t_i)|^2_{V_h^*},
\end{aligned}
\tag{6.18}
$$

with $\nu > 0$, using Cauchy-Schwarz and Cauchy's inequalities, and owing to (2) in Assumption 1, to (2) in Assumption 7 and to Assumption 8.

From (6.15), (6.16), (6.17) and (6.18), we obtain the following estimate for the last term in (6.11):

$$
\begin{aligned}
&|L_{h\varepsilon,i}w(t_i)\varepsilon + \varphi(t_i)|^2_{H_h}\\
&\le (1+\nu)M^2C_h^2\varepsilon|w(t_i)|^2_{V_h}\varepsilon\\
&\quad + \Big(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\Big)M^2C_h^2\Big(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\Big)^2\\
&\quad + \Big(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\Big)C_h^2|\varphi_1(t_i)|^2_{V_h^*} + \Big(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\Big)C_h^2|\varphi_2(t_i)|^2_{V_h^*}.
\end{aligned}
\tag{6.19}
$$

Putting estimates (6.12), (6.14) and (6.19) together and summing up, owing to Assumption 6, we have, for $j = 0, 1, \dots, n$,

$$
\begin{aligned}
|w(t_j)|^2_{H_h} \leq 2K \sum_{i=0}^{j-1} |w(t_i)|^2_{H_h}\varepsilon + ((1+\nu)M^2C_h^2\varepsilon - \lambda) \sum_{i=0}^{j-1} |w(t_i)|^2_{V_h}\varepsilon \\
+ M^2C^2\Big(\Big(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\Big)C_h^2\varepsilon + \frac{3}{\lambda}\Big)\sum_{i=0}^{j-1}\varepsilon^{2\delta+1} \\
+ \Big(\Big(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\Big)C_h^2\varepsilon + \frac{3}{\lambda}\Big)\sum_{i=0}^{j-1}\frac{1}{\varepsilon}|\varphi_1(t_i)|^2_{V_h^*} \\
+ \Big(\Big(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\Big)C_h^2\varepsilon + \frac{3}{\lambda}\Big)\sum_{i=0}^{j-1}\frac{1}{\varepsilon}|\varphi_2(t_i)|^2_{V_h^*}.
\end{aligned}
\tag{6.20}
$$

As we assume that there is a constant p such that $M^2C_h^2\varepsilon \leq p < \lambda$, we have that, for ν sufficiently small,

$$
(1+\nu)M^2C_h^2\varepsilon - \lambda \leq (1+\nu)p - \lambda < 0.
$$

Then, from (6.20),

$$
\begin{aligned}
&|w(t_j)|^2_{H_h} + (\lambda - (1+\nu)p)\sum_{i=0}^{j-1}|w(t_i)|^2_{V_h}\varepsilon \\
&\leq 2K\sum_{i=0}^{j-1}|w(t_i)|^2_{H_h}\varepsilon + L\,\varepsilon^{2\delta} + L\sum_{i=0}^{n-1}\frac{1}{\varepsilon}\Big|L_{h\varepsilon,i}u_h(t_i)\varepsilon - \int_{t_i}^{t_{i+1}} L_h(s)u_h(t_i)ds\Big|^2_{V_h^*} \\
&\quad + L\sum_{i=0}^{n-1}\frac{1}{\varepsilon}\Big|f_{h\varepsilon,i}\varepsilon - \int_{t_i}^{t_{i+1}} f_h(s)ds\Big|^2_{V_h^*},
\end{aligned}
$$

where $L := (3\mu\nu M^2 + \lambda((1+\mu)\nu + \mu(\mu\nu+3))p)/\lambda\mu\nu M^2$. Estimates (1) and (2) are obtained following the same steps as in Theorem 9. □

Next result follows immediately from Theorem 10.

Corollary 4. *Assume that the hypotheses of Theorem 10 are satisfied. Denote by $u_h(t)$ the unique solution of problem (6.1) in Theorem 1, and by $v_{h\varepsilon,j}$, with $j = 0, 1, \dots, n$, the unique solution of problem (6.2). If there exists a constant N independent of ε such that*

$$
\Big|L_{h\varepsilon,j}u_h(t_j) - \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} L_h(s)u_h(t_j)ds\Big|^2_{V_h^*} + \Big|f_{h\varepsilon,j} - \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} f_h(s)ds\Big|^2_{V_h^*} \leq N\varepsilon^{2\delta},
$$

for $j = 0, 1, \dots, n-1$, then

$$
\max_{0\leq j\leq n}|v_{h\varepsilon,j} - u_h(t_j)|^2_{H_h} \leq N\varepsilon^{2\delta} \quad \text{and} \quad \sum_{j=0}^{n}|v_{h\varepsilon,j} - u_h(t_j)|^2_{V_h}\varepsilon \leq N\varepsilon^{2\delta}.
$$

Remark 15. Similarly to what we observed in Remark 12 for the case of the implicit scheme, if operators L_h and f_h in (6.1) have the particular time-discretization, for all $z \in V_h$,

$$
\bar{L}_{h\varepsilon}(t_j)z := \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} L_h(s)zds \quad \text{and} \quad \bar{f}_{h\varepsilon}(t_j) := \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} f_h(s)ds,
$$

$j = 0, 1, \ldots, n-1$, respectively, the estimates in Theorem 10 have a simpler form. Equally, no assumption needs to be made over the discrete operators, and Assumption 7 is satisfied under Assumption 1.

7. Numerical approximation in space and time

In Section 3, we proceeded to the space-discretization of second-order parabolic PDE problem (2.2). We took its version (3.1), discrete in space:

$$L_h u - u_t + f_h = 0 \quad \text{in} \quad Q(h), \qquad u(0,x) = g_h(x) \quad \text{in} \quad Z_h^d,$$

where $Q(h) = [0,T] \times Z_h^d$, with $T \in (0,\infty)$, Z_h^d is a h-grid on $\mathbb{R}^d$ and L_h the discrete operator

$$L_h(t,x) = a^{ij}(t,x)\partial_j^- \partial_i^+ + b^i(t,x)\partial_i^+ + c\,(t,x),$$

and set a suitable space-discrete framework, considering the discrete weighted spaces $l^{0,2}(r)$ and $l^{1,2}(r,\rho)$, in order to handle the unbounded data. We then showed that this discrete framework is a particular case of the general framework presented in Section 2 (in particular, Assumption 1 is satisfied).

Let us now consider, for the time-discretization of the second-order problem (3.1), both the implicit scheme

$$\Delta^- v_{i+1} = L_{h\varepsilon,i+1} v_{i+1} + f_{h\varepsilon,i+1} \quad \text{for} \quad i = 0, 1, \ldots, n-1, \quad v_0 = g_h \tag{7.1}$$

and the explicit scheme

$$\Delta^+ v_i = L_{h\varepsilon,i}\, v_i + f_{h\varepsilon,i} \quad \text{for} \quad i = 0, 1, \ldots, n-1, \quad v_0 = g_h. \tag{7.2}$$

From the above, the results obtained in Sections 5 and 6 under general frameworks, still hold for this particular problem.

It remains only to determine the rate of convergence when the discretization is considered both in space and time. We will prove that the approximation is $(\varepsilon^\delta + h)$-accurate.

We first establish the result for the case where the implicit scheme is used for the time-discretization.

Theorem 11. *Assume that the hypotheses of Theorems 5 and 8 are satisfied. Denote by u the solution of (2.2) in Theorem 2, u_h the solution of (3.1) in Theorem 3, and $v_{h\varepsilon,j}$, $j = 0, 1, \ldots, n$, the solution of (7.1) in Theorem 6. Then*

$$\begin{aligned}
&\max_{0\le j\le n} |v_{h\varepsilon,j} - u(t_j)|^2_{l^{0,2}(r)} + \sum_{0\le j\le n} |v_{h\varepsilon,j} - u(t_j)|^2_{l^{1,2}(r,\rho)}\,\varepsilon \\
&\le N\Big(\varepsilon^{2\delta} + h^2 \int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)} dt\Big) + N\Big(\sum_{1\le j\le n} \frac{1}{\varepsilon}\Big|L_{h\varepsilon,j} u_h(t_j)\varepsilon - \int_{t_{j-1}}^{t_j} L_h(s) u_h(t_j)\,ds\Big|^2_{l^{0,2}(r)} \\
&\quad + \sum_{1\le j\le n} \frac{1}{\varepsilon}\Big|f_{h\varepsilon,j}\varepsilon - \int_{t_{j-1}}^{t_j} f_h(s)\,ds\Big|^2_{l^{0,2}(r)} + |g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\,dt\Big)
\end{aligned}$$

with N a constant independent of h and ε.

Proof. Let us consider separately the two terms in the sum we want to estimate.

For the first term,

$$\begin{aligned}
&\max_{0\le j\le n} |v_{h\varepsilon,j} - u(t_j)|^2_{l^{0,2}(r)} \\
&\le 2 \max_{0\le j\le n} |v_{h\varepsilon,j} - u_h(t_j)|^2_{l^{0,2}(r)} + 2 \sup_{0\le t\le T} |u(t) - u_h(t)|^2_{l^{0,2}(r)},
\end{aligned} \tag{7.3}$$

and the desired estimate is obtained immediately by using Theorems 5 and 8.

For the second term, we have

$$\sum_{j=0}^{n} |v_{h\varepsilon,j} - u(t_j)|^2_{l^{1,2}(r,\rho)}\,\varepsilon \le 2\sum_{j=0}^{n} |v_{h\varepsilon,j} - u_h(t_j)|^2_{l^{1,2}(r,\rho)}\,\varepsilon + 2\sum_{j=0}^{n} |u_h(t_j) - u(t_j)|^2_{l^{1,2}(r,\rho)}\,\varepsilon. \tag{7.4}$$

Let us determine an estimate for the second term in (7.4).

Denote $|\ \ |_{l^{1,2}(r,\rho)} := |\ \ |_1$. We have that

$$\Big|\sum_{j=0}^{n} |u_h(t_j) - u(t_j)|_1^2\,\varepsilon - \int_0^T |u_h(t) - u(t)|_1^2\,dt\Big| = |u_h(t_0)-u(t_0)|_1^2\varepsilon + \Big|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\big(|u_h(t_{j+1})-u(t_{j+1})|_1^2 - |u_h(s)-u(s)|_1^2\big)ds\Big|. \tag{7.5}$$

For the integral in (7.5), using Cauchy's inequality and Assumption 6, we have

$$\begin{aligned}
&\int_{t_j}^{t_{j+1}}\big(|u_h(t_{j+1}) - u(t_{j+1})|_1^2 - |u_h(s) - u(s)|_1^2\big)ds\\
&\le 2|u_h(t_{j+1}) - u(t_{j+1})|_1\int_{t_j}^{t_{j+1}}\big(|u_h(t_{j+1}) - u(t_{j+1})|_1 - |u_h(s) - u(s)|_1\big)ds\\
&\le \lambda\varepsilon|u_h(t_{j+1})-u(t_{j+1})|_1^2+\frac{1}{\lambda\varepsilon}\Big(\int_{t_j}^{t_{j+1}}\big(|u_h(t_{j+1})-u(t_{j+1})|_1-|u_h(s)-u(s)|_1\big)ds\Big)^2\\
&\le \lambda\varepsilon|u_h(t_{j+1})-u(t_{j+1})|_1^2+\frac{\varepsilon}{\lambda}N\varepsilon^{2\delta},
\end{aligned} \tag{7.6}$$

with $\lambda > 0$.

From (7.5) and (7.6),

$$\begin{aligned}
&\Big|\sum_{j=0}^{n} |u_h(t_j) - u(t_j)|_1^2\,\varepsilon - \int_0^T |u_h(t) - u(t)|_1^2\,dt\Big|\\
&\le |u_h(t_0) - u(t_0)|_1^2\varepsilon + \lambda\sum_{j=0}^{n-1}|u_h(t_{j+1}) - u(t_{j+1})|_1^2\varepsilon + N\varepsilon^{2\delta}\\
&= (1-\lambda)|u_h(t_0) - u(t_0)|_1^2\varepsilon + \lambda\sum_{j=0}^{n}|u_h(t_{j+1}) - u(t_{j+1})|_1^2\varepsilon + N\varepsilon^{2\delta}\\
&\le \lambda\sum_{j=0}^{n}|u_h(t_{j+1}) - u(t_{j+1})|_1^2\varepsilon + N\varepsilon^{2\delta}
\end{aligned}$$

and, for $0 < \lambda < 1$ we finally obtain

$$\sum_{j=0}^{n} |u_h(t_j) - u(t_j)|_1^2\,\varepsilon \le N\int_0^T |u_h(t) - u(t)|_1^2\,dt + N\varepsilon^{2\delta}. \tag{7.7}$$

From (7.4) and (7.7), the desired estimate is obtained immediately owing to Theorem 5. The result is proved. ∎

Next result follows immediately from Theorem 11.

Corollary 5. *Assume that the hypotheses of Theorem 11 are satisfied, and denote by u the solution of (2.2) in Theorem 2, u_h the solution of (3.1) in Theorem 3, and $v_{h\varepsilon,j}$, $j = 0, 1, \dots, n$, the solution of (7.1) in Theorem 6. If there exists a constant N independent of h and ε such that*

$$\big|L_{h\varepsilon,j}u_h(t_j) - \frac{1}{\varepsilon}\int_{t_{j-1}}^{t_j} L_h(s)u_h(t_j)ds\big|^2_{l^{0,2}(r)} + \big|f_{h\varepsilon,j} - \frac{1}{\varepsilon}\int_{t_{j-1}}^{t_j} f_h(s)ds\big|^2_{l^{0,2}(r)} \le N\varepsilon^{2\delta},$$

for $j = 1, 2, \dots, n$, and

$$|g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\,dt \le Nh^2\Big(|g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)}dt\Big)$$

then

$$\max_{0\le j\le n} |v_{h\varepsilon,j} - u(t_j)|^2_{l^{0,2}(r)} + \sum_{0\le j\le n} |v_{h\varepsilon,j} - u(t_j)|^2_{l^{1,2}(r,\rho)}\,\varepsilon$$
$$\le N\varepsilon^{2\delta} + Nh^2\Big(\int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)}dt + |g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)}dt\Big).$$

Now, we determine the rate of convergence, in the case where the explicit scheme is used for the discretization in time. The proof is the same as for Theorem 11.

Theorem 12. *Let the hypotheses of Theorems 5 and 10 be satisfied. Denote by u the solution of (2.2) in Theorem 2, u_h the solution of (3.1) in Theorem 3, and $v_{h\varepsilon,j}$, $j = 0, 1, \dots, n$, the solution of (7.2). Then*

$$\max_{0\le j\le n} |v_{h\varepsilon,j} - u(t_j)|^2_{l^{0,2}(r)} + \sum_{0\le j\le n} |v_{h\varepsilon,j} - u(t_j)|^2_{l^{1,2}(r,\rho)}\,\varepsilon$$
$$\le N\Big(\varepsilon^{2\delta} + h^2\int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)}dt\Big) + N\Big(\sum_{j=0}^{n-1}\frac{1}{\varepsilon}\big|\,L_{h\varepsilon,j}u_h(t_j)\varepsilon - \int_{t_j}^{t_{j+1}} L_h(s)u_h(t_j)ds\big|^2_{l^{0,2}(r)}$$
$$+ \sum_{j=0}^{n-1}\frac{1}{\varepsilon}\big|f_{h\varepsilon,j}\varepsilon - \int_{t_j}^{t_{j+1}} f_h(s)ds\big|^2_{l^{0,2}(r)} + |g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\,dt\Big)$$

with N a constant independent of h and ε.

Finally, we state a corollary as immediate consequence of Theorem 12.

Corollary 6. *Assume that the hypotheses of Theorem 12 are satisfied, and denote by u the solution of (2.2) in Theorem 2, u_h the solution of (3.1) in Theorem 3, and $v_{h\varepsilon,j}$, $j = 0, 1, \dots, n$, the solution of (7.2). If there exists a constant N independent of h and ε such that*

$$\big|L_{h\varepsilon,j}u_h(t_j) - \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} L_h(s)u_h(t_j)ds\big|^2_{l^{0,2}(r)} + \big|f_{h\varepsilon,j} - \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} f_h(s)ds\big|^2_{l^{0,2}(r)} \le N\varepsilon^{2\delta},$$

for $j = 0, 1, \dots, n-1$ and

$$|g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\,dt \le Nh^2\Big(|g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)}dt\Big)$$

then

$$\max_{0\leq j\leq n}|v_{h\varepsilon,j}-u(t_j)|^2_{l^{0,2}(r)}+\sum_{0\leq j\leq n}|v_{h\varepsilon,j}-u(t_j)|^2_{l^{1,2}(r,\rho)}\,\varepsilon$$
$$\leq N\varepsilon^{2\delta}+Nh^2\Big(\int_0^T|u(t)|^2_{W^{m+3,2}(r,\rho)}dt+|g|^2_{W^{m,2}(r,\rho)}+\int_0^T|f(t)|^2_{W^{m-1,2}(r,\rho)}dt\Big).$$

8. Final remarks

In this article, we investigated the numerical approximation of general second-order linear parabolic PDEs, in the framework of the variational approach.

By considering a suitable class of weighted Sobolev spaces, and its zero and first-order discrete versions, we could deal with the growth of the PDE coefficients with respect to the space variables. The discretization in time was pursued within an abstract framework and then specified for the case of a second-order parabolic PDE problem. The rate of convergence of the approximation was estimated.

The numerical approximation was studied under the strong assumption that the PDE is nondegenerate. The approach of the degenerate case is an immediate continuation of the present study. For this approach, the results obtained in [11, 12] will play a central role.

In a closing remark, we emphasize that this article was intended to be a first theoretical effort at the numerical approximation of generalized solutions of parabolic equations. We understand that the accuracy of the approximation we produced is not good enough for practical purposes. Possible further research directions include: the use of splitting-up methods (see [10]), following Richardson's idea to accelerate numerical schemes; and also the use of techniques reducing the volume of computational work (e.g., sparse grids), in order to deal with the computational challenge posed by the possible high dimensionality of the problem.

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