SPACE DISCRETIZATION OF PDES WITH UNBOUNDED COEFFICIENTS CONNECTED TO OPTION PRICING – THE CASE OF ONE SPACIAL DIMENSION

F. F. GONÇALVES AND M. R. GROSSINHO

Abstract. We study the space discretization of the Cauchy problem for a second order linear parabolic PDE, with one spacial dimension and unbounded time and space-dependent coefficients. The equation free term and the initial data are also allowed to grow. Under a nondegeneracy assumption, we consider the PDE solvability in the framework of the variational approach, and approximate in space the PDE problem's weak solution with the use of finite-difference methods. The rate of convergence is estimated.

1. Introduction

In this article, we make use of finite-difference methods to approximate in space the weak solution of the Cauchy problem

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ in } \mathbb{R}, \tag{1.1}$$

where $Q = [0,T] \times \mathbb{R}$, with T a positive constant, L is the partial differential operator with real coefficients

$$L(t,x) = a(t,x)\frac{\partial^2}{\partial x^2} + b(t,x)\frac{\partial}{\partial x} + c(t,x),$$

for each $t \in [0,T]$ uniformly elliptic in the space variable, and f and g are given real-valued functions. We allow the growth in space of the first and second-order coefficients in L (linear and quadratic growth, respectively), and of the data f and g (polynomial growth).

We point out that by considering problem (1.1), with one dimension in space, we can prove a convergence result stronger than the corresponding result for the multidimensional case (see, e.g, [7]).

Linear parabolic PDE problems arise in Financial Mathematics and in Mathematical Physics. We are mainly motivated by the application to Finance, namely to a class of stochastic option pricing models , comprising the non path-dependent options, with fixed exercise (see, e.g., [12]).

Consider a general version of Black-Scholes stochastic model for a financial option of European type, where the underlying asset appreciation rate and volatility are taken time and space-dependent. Under this model, pricing an option can be reduced, making use of a Feynman-Kăc type formula, to solving the Cauchy problem

---

2000 *Mathematics Subject Classification.* 65M06, 65M12, 65M15.

Key words and phrases. Cauchy problem, parabolic partial differential equations, unbounded coefficients; finite-difference methods; option pricing.

Fundação para a Ciência e Tecnologia-FCT (SFRH/BPD/35734/2008) to F.F.G.

Fundação para a Ciência e Tecnologia-FCT (FEDER/POCI 2010) to M.R.G.

(with terminal condition) for a degenerate parabolic PDE, with unbounded coefficients and null term (see, e.g., [12]). Therefore, instead of approximating the option price with probabilistic methods, we can approximate numerically the solution of the corresponding PDE problem.

In this article, we deal with the challenge posed by the unboundedness of the PDE coefficients, under the strong assumption that the PDE does not degenerate.

When problem (1.1) is considered in connection with the Black-Scholes modelling of a financial option[1], we see that the general case where the asset appreciation rate and volatility are taken time and space-dependent is covered, with the growth of underlying SDE coefficients being appropriately matched. Also, by letting the initial data g non-specified, a large class of pay-off functions can be considered in the underlying financial derivative modelling. The inclusion of the free term f further improves generality.

There is a strong link between the numerical methods and possible approximation results and the theory on the solvability of the PDEs. We make use of the L^2 theory of solvability of linear PDEs in weighted Sobolev spaces. In particular, we consider the PDE solvability in the deterministic one spacial dimension special case of a class of weighted Sobolev spaces introduced by O. G. Purtukhia [15, 16, 17, 18], and further generalized by I. Gyöngy and N. V. Krylov [8], for the treatment of linear SPDEs. By considering discrete versions of these spaces, we construct a suitable discretized framework and investigate the PDE approximation.

The finite-difference method for approximating PDE is a mature area of research.[2] We refer to [20] for the numerical approximation study of the Cauchy problem for general multidimensional linear parabolic PDEs of order $m \geq 2$, with bounded time and space-dependent coefficients. This study is pursued in the framework of the classical approach.

Also, there has been a long and extensive research on the application of finite-difference methods to financial option pricing.[3]

Most studies concerning the numerical approximation of PDE problems in Finance consider the particular case where the PDE coefficients are constant (see, e.g., [1, 2, 5, 19]). This occurs in Black-Scholes models (in one or several dimensions), when the underlying asset appreciation rate and volatility are taken constant. The simpler PDE with constant coefficients is obtained after a standard change of variables (see, e.g., [12] for the one-dimensional case, and [6] for the mutidimensional case).

In [14], a space-time adaptive finite-difference method is developed for the approximation of a multidimensional PDE problem, corresponding to a version of Black-Scholes model where the vector of asset appreciation rates and volatility matrix are taken variable but only with respect to the time variable. The difficulty coming from the unboundedness of the PDE coefficients is not considered, as the discretization is made after a spatial domain truncation.

---

[1] Instead of the terminal-value formulation arising from the financial model, we consider the more standard initial-value formulation. Clearly, one problem can be transformed in the other by a simple change of the time variable.

[2] See [20] for a brief history of the finite-difference method, and also for the references of the original publications and further major contributions.

[3] The references for the seminal applications of finite differences to financial problems and further major research can be tracked down in the review paper [3].

With the present article, we aim to contribute to the systematic study of the numerical approximation of problem (1.1). We consider the problem in the framework of the variational approach, and impose weak regularity over the operator's coefficients and the data f and g. In order to facilitate the study, we make use of basic one-step finite-difference schemes.

In [7], the same approach is utilized for the numerical approximation of the more general multidimensional version of problem (1.1).[4] In the present article, by considering the special case of one dimension in space, a stronger convergence result is proved. In particular, the same accuracy is obtained under regularity assumptions over the solution of (1.1) weaker than those required in [7] for the corresponding convergence result.

We summarize the content of the article. In Section 2, we state well-known facts on the solvability of linear PDEs under a general framework, and introduce a suitable class of weighted Sobolev spaces. In Section 3, we discretize in space problem (1.1), with the use of finite-difference schemes, and introduce discrete versions of the weighted Sobolev spaces. By showing that the discrete framework we set is a particular case of the general framework presented in the previous Section, we deduce an existence and uniqueness result for the generalized solution of the discretized problem. In Section 4, we prove that this discrete solution approximates the exact generalized solution of problem (1.1), and determine the rate of convergence. In Section 5, we make some final comments.

2. Preliminaries and classical results

We establish some facts on the solvability of PDEs under a general framework.

Let V be a reflexive separable Banach space embedded continuously and densely into a Hilbert space H with inner product $(\ , \ )$. Then H^*, the dual of H, is also continuously and densely embedded into V^*, the dual of V. Let us use the notation $\langle \ , \ \rangle$ for the duality. Let H^* be identified with H in the usual way, by the help of the inner product. Then we have the so called normal triple $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$, with continuous and dense embeddings.

Consider the Cauchy problem for an evolution equation

$$L(t)u(t) - \frac{\partial u(t)}{\partial t} + f(t) = 0, \quad u(0) = g, \tag{2.1}$$

where $L(t)$ and $\partial/\partial t$ are linear operators from V to V^*, $f(t) \in V^*$, for every $t \in [0,T]$ with $T \in (0,\infty)$, and $g \in H$.

We assume the continuity and coercivity of operator $L(t)$, and impose some regularity over the data f and g:

Assumption 1. There exist constants $\lambda > 0$, K, M and N such that

(1) $\langle L(t)v, v\rangle + \lambda |v|_V^2 \leq K|v|_H^2, \quad \forall v \in V$ and $\forall t \in [0,T]$;
(2) $|L(t)v|_{V^*} \leq M|v|_V, \quad \forall v \in V$ and $\forall t \in [0,T]$;
(3) $\int_0^T |f(t)|_{V^*}^2 dt \leq N$ and $|g|_H \leq N$.

We define the generalized solution of problem (2.1).

[4] [7] also presents the approximation in time of the multidimensional version of the problem (1.1).

Definition 1. We say that $u \in C([0,T];H)$ is a generalized solution of (2.1) on $[0,T]$ if

1. $u \in L^2([0,T];V)$;
2. $(u(t),v) = (g,v) + \int_0^t \langle L(s)u(s), v\rangle ds + \int_0^t \langle f(s), v\rangle ds$
   holds for every $t \in [0,T]$, $v \in V$.

Notation. We denote by $C([0,T];W)$, with W a Banach space, the set of all continuous W-valued functions on $[0,T]$. The notation $L^2([0,T];W)$ stands for the set of L^2 W-valued functions on $[0,T]$.

Under Assumption 1, problem (2.1) has a unique generalized solution.

Theorem 1. *Under (1)–(3) in Assumption 1, problem (2.1) has a unique generalized solution on* $[0,T]$. *Moreover*

$$\sup_{t\in[0,T]} |u(t)|_H^2 + \int_0^T |u(t)|_V^2 dt \le N\Big(|g|_H^2 + \int_0^T |f(t)|_{V^*}^2 dt\Big),$$

where N is a constant.

The above well-known result is a special case of a more general one proved in [13] for nonlinear evolution equations.

Let us now consider the particular PDE problem

$$Lu - u_t + f = 0 \ \text{ in } \ Q, \quad u(0,x) = g(x) \ \text{ in } \ \mathbb{R}, \tag{2.2}$$

where L is the second-order operator with real coefficients

$$L(t,x) = a(t,x)\frac{\partial^2}{\partial x^2} + b(t,x)\frac{\partial}{\partial x} + c(t,x), \tag{2.3}$$

$Q = [0,T]\times\mathbb{R}$, with $T \in (0,\infty)$, and f and g are given functions. We allow the growth, in the space variables, of the coefficient functions a and b, and of the free data f and g.

To set the framework for problem (2.2), we introduce a suitable class of weighted Sobolev spaces.[5]

Let U be a domain in $\mathbb{R}$, i.e., an open subset of $\mathbb{R}$. Let $r > 0$, $\rho > 0$ be smooth functions in U and $m \ge 0$ an integer. The weighted Sobolev space $W^{m,2}(r,\rho)(U)$ is the Banach Space of all locally integrable functions $v : U \to \mathbb{R}$ such that for each integer $\alpha \ge 0$, with $\alpha \le m$, $D^\alpha v$ exists in the weak sense, and $|v|_{W^{m,2}(r,\rho)(U)} := (\sum_{\alpha\le m}\int_U r^2|\rho^\alpha D^\alpha v|^2 dx)^{1/2}$ is finite. Endowed with the inner product $(v,w)_{W^{m,2}(r,\rho)(U)} := \sum_{\alpha\le m}\int_U r^2\rho^{2\alpha} D^\alpha v D^\alpha w\, dx$, for all $v,w \in W^{m,2}(r,\rho)(U)$, which generates the norm, $W^{m,2}(r,\rho)(U)$ is a Hilbert space.

Remark 1. Setting the weight functions $r = \rho = 1$, for all $x \in U$, we obtain the particular case of the Sobolev spaces $W^{m,2}(U)$.

Notation. In the sequel, when $U = \mathbb{R}$ the argument in the function space notation is dropped. For instance, we denote $W^{m,2}(r,\rho)(\mathbb{R}) =: W^{m,2}(r,\rho)$.

We make some assumptions on the behaviour of the weight functions r and ρ (see [8]).

[5] We refer to [8] for a complete description of this class of spaces.

Assumption 2. Let $m \geq 0$ be an integer and $r > 0$ and $\rho > 0$ smooth functions on $\mathbb{R}$. There exists a constant K such that,

(1) $|D^\alpha \rho| \leq K\rho^{1-\alpha}$ for all α such that $\alpha \leq m-1$ if $m \geq 2$;
(2) $|D^\alpha r| \leq K\dfrac{r}{\rho^\alpha}$ for all α such that $\alpha \leq m$;
(3) $\sup\limits_{|x-y|<\varepsilon} \left(\dfrac{r(x)}{r(y)} + \dfrac{\rho(x)}{\rho(y)}\right) = K$ for some $\varepsilon > 0$, $x, y \in \mathbb{R}$.

Example 1. The following functions (taken from [8], citing [15]) satisfy $(1)-(3)$ in Assumption 2:

(1) $r(x) = (1+|x|^2)^\beta$, $\beta \in \mathbb{R}$; $\rho(x) = (1+|x|^2)^\gamma$, $\gamma \leq \frac{1}{2}$;
(2) $r(x) = \exp(\pm(1+|x|^2)^\beta)$, $0 \leq \beta \leq \frac{1}{2}$; $\rho(x) = (1+|x|^2)^\gamma$, $\gamma \leq \frac{1}{2} - \beta$;
(3) $r(x) = (1+|x|^2)^\beta$, $\beta \in \mathbb{R}$; $\rho(x) = \ln^\gamma(2+|x|^2)$, $\gamma \in \mathbb{R}$;
(4) $r(x) = (1+|x|^2)^\beta \ln^\mu(2+|x|^2)$, $\beta \geq 0$, $\mu \geq 0$; $\rho(x) = (1+|x|^2)^\gamma$, $\gamma \leq \frac{1}{2}$;
(5) $r(x) = (1+|x|^2)^\beta \ln^\mu(2+|x|^2)$, $\beta \geq 0$, $\mu \geq 0$; $\rho(x) = \ln^\gamma(2+|x|^2)$, $\gamma \geq 0$;
(6) $\rho(x) = \exp(-(1+|x|^2)^\gamma)$, $\gamma \geq 0$; each weight function $r(x)$ in examples $(1)-(5)$.

Now, we switch point of view and consider the functions $w : Q \to \mathbb{R}$ as functions on $[0,T]$ with values in $\mathbb{R}^\infty$ such that, for all $t \in [0,T]$, $w(t) := \{w(t,x) : x \in \mathbb{R}\}$.

We impose a coercivity condition over the operator (2.3), and make assumptions on the growth and regularity of the operator's coefficients and also on the regularity of the free data f and g (see [8]):

Assumption 3. Let $r > 0$ and $\rho > 0$ be smooth functions on $\mathbb{R}$ and $m \geq 0$ an integer.

(1) There exists a constant $\lambda > 0$ such that $a(t,x) \geq \lambda\rho^2(x)$, for all $t \geq 0$, $x \in \mathbb{R}$;
(2) The coefficients in L and their derivatives in x up to the order m are measurable functions in $[0,T] \times \mathbb{R}$ such that
$$|D_x^\alpha a| \leq K\rho^{2-\alpha} \ \forall \alpha \leq m \vee 1, \ |D_x^\alpha b| \leq K\rho^{1-\alpha}, \ |D_x^\alpha c| \leq K \ \forall \alpha \leq m,$$
for any $t \in [0,T]$, $x \in \mathbb{R}$, with K a constant and D_x^α denoting the α^{th} partial derivative operator with respect to x;
(3) $f \in L^2([0,T]; W^{m-1,2}(r,\rho))$ and $g \in W^{m,2}(r,\rho)$.

Remark 2. For $m=0$ we use the notation $W^{m-1,2}(r,\rho) = W^{-1,2}(r,\rho) := (W^{1,2}(r,\rho))^*$, where $(W^{1,2}(r,\rho))^*$ is the dual of $W^{1,2}(r,\rho)$.

We define the generalized solution of problem (2.2).

Definition 2. We say that $u \in C([0,T]; W^{0,2}(r,\rho))$ is a generalized solution of (2.2) on $[0,T]$ if

(1) $u \in L^2([0,T]; W^{1,2}(r,\rho))$;
(2) For every $t \in [0,T]$
$$\begin{aligned}(u(t),\varphi) = (g,\varphi) + \int_0^t \Big\{ &- \big(a(s)D_x u(s), D_x\varphi\big) \\ &+ \big(b(s)D_x u(s) - D_x a(s) D_x u(s), \varphi\big) \\ &+ \big(c(s)u(s),\varphi\big) + \langle f(s),\varphi\rangle \Big\} ds\end{aligned}$$
holds for all $\varphi \in C_0^\infty$.

Notation. The notation (,) in the above definition stands for the inner product in $W^{0,2}(r,\rho)$. C_0^∞ denotes the set of all infinitely differentiable functions on $\mathbb{R}$ with compact support.

Remark 3. Alternatively to the infinite differentiability of φ in (2), it can be required that $\varphi \in W^{1,2}(r,\rho)$.

Finally, we state the existence and uniqueness of the solution of problem (2.2).

Theorem 2. *Under (1)–(2) in Assumption 2, with $m+1$ in place of m, and (1)–(3) in Assumption 3, problem (2.2) admits a unique generalized solution u on $[0,T]$. Moreover $u \in C([0,T];W^{m,2}(r,\rho)) \cap L^2([0,T];W^{m+1,2}(r,\rho))$ and*

$$\sup_{0\le t\le T} |u(t)|^2_{W^{m,2}(r,\rho)} + \int_0^T |u(t)|^2_{W^{m+1,2}(r,\rho)} dt \le N\Big(|g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)} dt \Big),$$

with N a constant.

The above result can be obtained from the general result in abstract spaces (Theorem 1) by using the suitable triple of spaces (see [8]).

3. The discrete framework

We now proceed to the discretization of problem (2.2) in the space-variable. We set a suitable discrete framework with the use of a finite-difference scheme and, by showing that the discretized problem can be cast into the general problem (2.1), we prove an existence and uniqueness result for the discrete problem's generalized solution.

We define the h-grid on $\mathbb{R}$, with $h \in (0,1]$

$$Z_h = \big\{ x \in \mathbb{R} : x = nh, \ \ n = 0, \pm 1, \pm 2, \dots \big\}.$$

Denote

$$\partial^+ u = \partial^+ u(t,x) = h^{-1}(u(t,x+h) - u(t,x))$$

and

$$\partial^- u = \partial^- u(t,x) = h^{-1}(u(t,x) - u(t,x-h)),$$

for every $x \in Z_h$, the forward and backward discrete differences in space, respectively. Define the discrete operator

$$L_h(t,x) = a(t,x)\partial^-\partial^+ + b(t,x)\partial^+ + c\,(t,x).$$

We consider the discrete problem

$$L_h u - u_t + f_h = 0 \ \text{ in } \ Q(h), \quad u(0,x) = g_h(x) \ \text{ in } \ Z_h, \tag{3.1}$$

where $Q(h) = [0,T] \times Z_h$, with $T \in (0,\infty)$, and f_h and g_h are functions such that $f_h : Q(h) \to \mathbb{R}$ and $g_h : Z_h \to \mathbb{R}$.

For functions $v : Z_h \to \mathbb{R}$, we introduce the discrete version of the weighted Sobolev space $W^{0,2}(r,\rho)$:

$$l^{0,2}(r) = \{ v : |v|_{l^{0,2}(r)} < \infty \},$$

where the norm $|v|_{l^{0,2}(r)}$ is defined by

$$|v|_{l^{0,2}(r)} = \Big(\sum_{x\in Z_h} r^2(x)|v(x)|^2 h \Big)^{1/2}.$$

Define the inner product

$$(v,w)_{l^{0,2}(r)} = \sum_{x\in Z_h} r^2(x)v(x)w(x)h,$$

for any $v,w \in l^{0,2}(r)$, which induces the above norm. Endowed with the inner product, the space $l^{0,2}(r)$ is clearly a Hilbert space.

For functions $v : Z_h \to \mathbb{R}$, we introduce also the discrete version of the weighted Sobolev space $W^{1,2}(r,\rho)$:

$$l^{1,2}(r,\rho) = \{v : |v|_{l^{1,2}(r,\rho)} < \infty\},$$

with the norm $|v|_{l^{1,2}(r,\rho)}$ defined by

$$|v|^2_{l^{1,2}(r,\rho)} = |v|^2_{l^{0,2}(r)} + |\rho\, \partial^+ v|^2_{l^{0,2}(r)}.$$

We endow $l^{1,2}(r,\rho)$ with the inner product, inducing the above norm,

$$(v,w)_{l^{1,2}(r,\rho)} = (v,w)_{l^{0,2}(r)} + (\rho\, \partial^+ v, \rho\, \partial^+ w)_{l^{0,2}(r)},$$

for v,w any functions in $l^{1,2}(r,\rho)$.

It can be easily checked that $l^{1,2}(r,\rho)$ is a reflexive and separable Banach space, continuously and densely embedded into the Hilbert space $l^{0,2}(r)$.[6] Thus, these discrete spaces can be cast in the normal triple of abstract spaces we considered in Section 2.

As for the solvability study of the PDE problem (2.2) in Section 2, we switch point of view and consider the functions $w : Q(h) \to \mathbb{R}$ as functions in $[0,T]$ with values in $\mathbb{R}^\infty$ defined by $w(t) = \{w(t,x) : x \in Z_h\}$, for all $t \in [0,T]$. For these functions, we consider the space $C([0,T]; l^{0,2}(r))$ of continuous $l^{0,2}(r)$-valued functions on $[0,T]$, and the spaces

$$L^2([0,T]; l^{m,2}(r,\rho)) = \Big\{ w : [0,T] \to l^{m,2}(r,\rho) : \int_0^T |w(t)|^2_{l^{m,2}(r,\rho)}\, dt < \infty \Big\},$$

with $m = 0,1$.

Notation. We identify $l^{0,2}(r,\rho)$ with $l^{0,2}(r)$.

Remark 4. Clearly, if $u \in C([0,T]; l^{0,2}(r))$ then $\sup_{t\in[0,T]} |u(t)|_{l^{0,2}(r)} < \infty$.

We make some assumptions over the regularity of the data f_h and g_h in (3.1).

Assumption 4. Let $r > 0$ be a smooth function on $\mathbb{R}$.

(1) $f_h \in L^2([0,T]; l^{0,2}(r))$;
(2) $g_h \in l^{0,2}(r)$.

Remark 5. In the above Assumption 4, (1) can be replaced for the weaker assumption $f_h \in L^2([0,T]; (l^{1,2}(r,\rho))^*)$, where $(l^{1,2}(r,\rho))^*$ denotes the dual space of $l^{1,2}(r,\rho)$.

Remark 6. We note that $|\partial^+ a| \le K\rho$ can be obtained from (2) in Assumption 3. In fact, by the mean value theorem,

$$|\partial^+ a(t,x)| = |h^{-1}(a(t,x+h) - a(t,x))| = \Big| \frac{\partial}{\partial x}\, a(t, x+\tau) \Big|,$$

for some τ such that $0 < \tau < h$. Thus $|(\partial/\partial x)\, a| \le K\rho$ implies $|\partial^+ a| \le K\rho$.

[6] We refer to [7], where this is proved for the more general case where $l^{0,2}(r)$ and $l^{1,2}(r,\rho)$ are spaces of real-valued functions on a d-dimensional grid.

We define the generalized solution of problem (3.1).

Definition 3. We say that $u \in C([0,T]; l^{0,2}(r)) \cap L^2([0,T]; l^{1,2}(r,\rho))$ is a generalized solution of (3.1) if, for all $t \in [0,T]$,

$$(u(t),\varphi) = (g_h,\varphi) + \int_0^t \Big\{ - \big(a(s)\partial^+ u(s), \partial^+\varphi\big) + \big(b(s)\partial^+ u(s) - \partial^+ a(s)\partial^+ u(s), \varphi\big) + \big(c(s)u(s),\varphi\big) + \langle f_h(s),\varphi\rangle \Big\} ds$$

holds for all $\varphi \in l^{1,2}(r,\rho)$.

Notation. In the above definition, (,) denotes the inner product in $l^{0,2}(r)$. We keep this convention in the sequel.

Next, we prove the existence and uniqueness of the generalized solution of discrete problem (3.1), and determine an estimate for the solution. With this result, we show that the numerical scheme is stable, i.e., informally, that the solution of the discrete problem remains bounded independently of the space-step h. The result is obtained as a consequence of Theorem 1, remaining only to show that, under the discrete framework we constructed, (1) − (2) in Assumption 1 hold.[7]

Theorem 3. *Under (1)–(2) in Assumption 3 and (1)–(2) in Assumption 4, problem (3.1) has a unique generalized solution u in $[0,T]$. Moreover*

$$\sup_{0\le t\le T} |u(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t)|^2_{l^{1,2}(r,\rho)} dt \le N\Big(|g_h|^2_{l^{0,2}(r)} + \int_0^T |f_h(t)|^2_{l^{0,2}(r)} dt \Big),$$

with N a constant independent of h.

Proof. Let $L_h(s) : l^{1,2}(r,\rho) \to (l^{1,2}(r,\rho))^*$. We define for all $\varphi,\psi \in l^{1,2}(r,\rho)$

$$\langle L_h(s)\psi,\varphi\rangle := -\,(a^{ij}(s)\partial_i^+\psi, \partial_j^+\varphi) + (b^i(s)\partial_i^+\psi - \partial_j^+ a^{ij}(s)\partial_i^+\psi, \varphi) + (c(s)\psi,\varphi).$$

It suffices to prove that the estimates

(1) $\exists K, \lambda > 0$ constants : $\langle L_h(s)\psi,\, \psi\rangle \le K|\psi|_{l^{0,2}(r)} - \lambda|\psi|_{l^{1,2}(r,\rho)}$
(2) $\exists K$ constant : $|\langle L_h(s)\psi,\, \varphi\rangle| \le K|\psi|_{l^{1,2}(r,\rho)} \cdot |\varphi|_{l^{1,2}(r,\rho)}$

hold for all $\varphi,\psi \in l^{1,2}(r,\rho)$.

For the first property, owing to (1) and (2) in Assumption 3, we have

$$\begin{aligned}
&\langle L_h(s)\psi,\psi\rangle \\
&= -\sum_{x\in Z_h} r^2 a(s)\partial^+\psi\,\partial^+\psi h + \sum_{x\in Z_h} r^2 (b(s) - \partial^+ a(s))\partial^+\psi\,\psi h + \sum_{x\in Z_h} r^2 c(s)\psi\,\psi h \\
&\le -\lambda \sum_{x\in Z_h} r^2|\rho\partial^+\psi|^2 h + 2K \sum_{x\in Z_h} r^2\rho|\partial^+\psi\,\psi| h + K \sum_{x\in Z_h} r^2|\psi|^2 h \qquad (3.2)\\
&= -\lambda|\rho\partial^+\psi|^2_{l^{0,2}(r)} + 2K \sum_{x\in Z_h} r^2\rho|\partial^+\psi\,\psi| h + K|\psi|^2_{l^{0,2}(r)},
\end{aligned}$$

[7] The result is proved in [7] for the more general multidimensional case. Here, we give that proof's particularization for the case of one dimension in space just in order to keep the article reasonably self-contained.

where the variable $x \in Z_h$ is omitted in the functions' argument. Applying Cauchy's inequality to the second term in estimate (3.2), we obtain

$$\begin{aligned}
&\langle L_h(s)\psi, \psi\rangle \\
&\le -\lambda|\rho\partial^+\psi|^2_{l^{0,2}(r)} + \varepsilon K \sum_{x\in Z_h} r^2|\rho\partial^+\psi|^2 h + \frac{K}{\varepsilon}\sum_{x\in Z_h} r^2|\psi|^2 h + K|\psi|^2_{l^{0,2}(r)} \\
&= -\lambda|\rho\partial^+\psi|^2_{l^{0,2}(r)} - \lambda|\psi|^2_{l^{0,2}(r)} + \varepsilon K|\rho\partial^+\psi|^2_{l^{0,2}(r)} + \frac{K}{\varepsilon}|\psi|^2_{l^{0,2}(r)} + (K+\lambda)|\psi|^2_{l^{0,2}(r)} \\
&\le -\lambda|\psi|^2_{l^{1,2}(r,\rho)} + K|\psi|^2_{l^{0,2}(r)},
\end{aligned}$$

with $\lambda > 0$, K constants, by taking ε sufficiently small. The first property is proved.

The second property follows from (2) in Assumption 3, using Cauchy-Schwarz inequality

$$\begin{aligned}
&\big|\langle L_h(s)\psi, \varphi\rangle\big| \\
&= \Big|-\sum_{x\in Z_h} r^2 a(s)\partial^+\psi\,\partial^+\varphi\, h + \sum_{x\in Z_h} r^2 b(s)\partial^+\psi\,\varphi h - \sum_{x\in Z_h} r^2\partial^+ a(s)\partial^+\psi\,\varphi\, h + \sum_{x\in Z_h} r^2 c(s)\psi\varphi h\Big| \\
&\le K\sum_{x\in Z_h} r^2|\rho^2\partial^+\psi\,\partial^+\varphi|\, h + K\sum_{x\in Z_h} r^2|\rho\partial^+\psi\,\varphi|\, h + K\sum_{x\in Z_h} r^2|\psi\,\varphi| h \\
&\le K|\rho\partial^+\psi|_{l^{0,2}(r)}|\rho\partial^+\varphi|_{l^{0,2}(r)} + K|\rho\partial^+\psi|_{l^{0,2}(r)}|\varphi|_{l^{0,2}(r)} + K|\psi|_{l^{0,2}(r)}|\varphi|_{l^{0,2}(r)} \\
&\le K|\psi|_{l^{1,2}(r,\rho)}\cdot|\varphi|_{l^{1,2}(r,\rho)},
\end{aligned}$$

where the same writing conventions are kept.

Owing to Theorem 1 the result follows. □

4. Approximation results

We now study the approximation properties of the numerical scheme (3.1). We begin by investigating the consistency of the numerical scheme, and prove that the discrete differences approximate the partial derivatives (with accuracy of order 1), under the assumption that the weights ρ are bounded from below by a positive constant.[8]

We stress that, by considering the special case of one dimension in space, we can prove a result stronger than the corresponding result in [7] for the more general multidimensional case (see Remark 4 below).

Theorem 4. *Assume that (1)–(3) in Assumption 2 are satisfied and that $\rho(x) \ge C$ on $\mathbb{R}$, with $C > 0$ a constant. Let $u(t) \in W^{2,2}(r,\rho)$, $v(t) \in W^{3,2}(r,\rho)$, for all $t \in [0,T]$. Then there exists a constant N independent of h such that*

$$(1)\quad \sum_{x\in Z_h} r^2(x)\Big|\frac{\partial}{\partial x}u(t,x) - \partial^+ u(t,x)\Big|^2\rho^2(x)h \le h^2 N|u(t)|^2_{W^{2,2}(r,\rho)},$$

$$(2)\quad \sum_{x\in Z_h} r^2(x)\Big|\frac{\partial^2}{\partial x^2}v(t,x) - \partial^-\partial^+ v(t,x)\Big|^2\rho^4(x)h \le h^2 N|v(t)|^2_{W^{3,2}(r,\rho)},$$

for all $t \in [0,T]$.

---

[8] Notice that this amounts to assume that the weights ρ are increasing functions of $|x|$, which is precisely the case we are studding.

Remark 7. Under the conditions of this theorem, function $u(t)$ (function $v(t)$) has a modification in x which is continuously differentiable in x up to the order 1 (up to the order 2), for every $t \in [0,T]$. Also, the partial derivatives in x up to the order 2 (up to the order 3) equal the weak derivatives a.e., for every $t \in [0,T]$. This can be proved by Sobolev's embedding of $W^{m,2}(B)$ into $C^n(\overline{B})$, for balls B in $\mathbb{R}$, if $m > \frac{1}{2} + n$, and by Morrey's inequality (see, e.g, [4, 13]). We consider these modifications in the theorem's proof.

Remark 8. When particularizing the hypotheses of the corresponding result in [7] to the case of one spatial dimension, we obtain the assumption that $u(t) \in W^{3,2}(r,\rho)$, $v(t) \in W^{4,2}(r,\rho)$, for all $t \in [0,T]$, which is stronger than assumed in Theorem 4.

Proof. (Theorem 4) Let us prove (1). Observe that the forward discrete difference can be written

$$\partial^+ u(t,x) = h^{-1}(u(t,x+h) - u(t,x)) = \int_0^1 \frac{\partial}{\partial x} u(t,x+hq)dq,$$

and

$$\begin{aligned}\Big(\frac{\partial}{\partial x} - \partial^+\Big)u(t,x) &= \int_0^1 \Big(\frac{\partial}{\partial x}u(t,x) - \frac{\partial}{\partial x} u(t,x+hq)\Big)dq \\ &= h\int_0^1 \int_0^1 q\frac{\partial^2}{\partial x^2} u(t,x+hqs)dsdq.\end{aligned} \tag{4.1}$$

From (4.1), using Jensen's inequality, we obtain

$$\begin{aligned}\Big|\Big(\frac{\partial}{\partial x} - \partial^+\Big)u(t,x)\Big|^2 &\le h^2 \int_0^1 \int_0^1 q^2 \Big|\frac{\partial^2}{\partial x^2} u(t,x+hqs)\Big|^2 dsdq \\ &= h\int_0^1 \int_0^{hq} q\Big|\frac{\partial^2}{\partial x^2} u(t,x+v)\Big|^2 dvdq \\ &\le h\int_0^1 qdq \int_0^h \Big|\frac{\partial^2}{\partial x^2} u(t,x+v)\Big|^2 dv \\ &= \frac{h}{2}\int_0^h \Big|\frac{\partial^2}{\partial x^2} u(t,x+v)\Big|^2 dv \\ &= \frac{h}{2}\int_x^{x+h} \Big|\frac{\partial^2}{\partial z^2} u(t,z)\Big|^2 dz.\end{aligned} \tag{4.2}$$

Observe also that from (4.2), using (3) in Assumption 2 we have, for any $\theta \in (0,1)$,

$$r^2(x)\Big|\Big(\frac{\partial}{\partial x} - \partial^+\Big)u(t,x)\Big|^2 \rho^2(x) \le hNr^2(x+\theta h)\rho^2(x+\theta h)\int_x^{x+h} \Big|\frac{\partial^2}{\partial z^2} u(t,z)\Big|^2 dz. \tag{4.3}$$

As, by the mean value theorem for integration, for some $\theta \in (0,1)$,

$$r^2(x+\theta h)\rho^2(x+\theta h)\int_x^{x+h} \Big|\frac{\partial^2}{\partial z^2} u(t,z)\Big|^2 dz = \int_x^{x+h} r^2(z)\Big|\frac{\partial^2}{\partial z^2} u(t,z)\Big|^2 \rho^2(z)dz, \tag{4.4}$$

from(4.3) and (4.4), using Hölder inequality, we obtain

$$
\begin{aligned}
&r^2(x)\Big|\Big(\frac{\partial}{\partial x}-\partial^+\Big)u(t,x)\Big|^2\rho^2(x)\\
&\le hN\int_x^{x+h} r^2(z)\Big|\frac{\partial^2}{\partial z^2}\,u(t,z)\Big|^2\rho^4(z)dz\cdot\sup_{z\in[x,x+h]}|\rho^{-2}(z)|\\
&\le hN\int_x^{x+h} r^2(z)\Big|\frac{\partial^2}{\partial z^2}\,u(t,z)\Big|^2\rho^4(z)dz,
\end{aligned}
\tag{4.5}
$$

owing to the hypotheses over the weights ρ.

Finally, summing up (4.5) over Z_h, we get

$$
\sum_{x\in Z_h} r^2(x)\Big|\Big(\frac{\partial}{\partial x}-\partial^+\Big)u(t,x)\Big|^2\rho^2(x)h\le h^2N|u(t)|^2_{W^{2,2}}(r,\rho),
$$

with N a constant independent of h, and (1) is proved.

We now prove (2). By writing the forward and backward discrete differences

$$
\partial^+v(t,x)=h^{-1}(v(t,x+h)-v(t,x))=\int_0^1\frac{\partial}{\partial x}\,v(t,x+hq)dq
$$

and

$$
\partial^-v(t,x)=h^{-1}(v(t,x)-v(t,x-h))=\int_0^1\frac{\partial}{\partial x}\,v(t,x-hs)ds,
$$

respectively, we have for the second-order discrete difference

$$
\begin{aligned}
\partial^-\partial^+v(t,x)=&\partial^-\int_0^1\frac{\partial}{\partial x}\,v(t,x+hq)dq\\
=&\int_0^1\Big(\frac{\partial}{\partial x}\int_0^1\frac{\partial}{\partial x}\,v(t,x+hq-hs)dq\Big)ds\\
=&\int_0^1\int_0^1\frac{\partial^2}{\partial x^2}\,v(t,x+h(q-s))dsdq,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\Big)v(t,x)\\
&=\int_0^1\int_0^1\Big(\frac{\partial^2}{\partial x^2}(t,x)-\frac{\partial^2}{\partial x^2}\,v(t,x+h(q-s))\Big)dsdq\\
&=h\int_0^1\int_0^1\int_0^1(q-s)\frac{\partial^3}{\partial x^3}\,v(t,x+hv(q-s))\,dvdsdq.
\end{aligned}
\tag{4.6}
$$

From (4.6), by Jensen's inequality,

$$
\begin{aligned}
&\Big|\Big(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\Big)v(t,x)\Big|^2\\
&\leq h^2\int_0^1\int_0^1\int_0^1|q-s|^2\Big|\frac{\partial^3}{\partial x^3}v(t,x+hv(q-s))\Big|^2 dvdsdq\\
&= h\int_0^1\int_0^1\int_0^{h(q-s)}(q-s)\Big|\frac{\partial^3}{\partial x^3}\,v(t,x+w)\Big|^2 dwdsdq\\
&\leq h\int_0^1\int_0^1\big|q-s\big|dsdq\int_0^h\Big|\frac{\partial^3}{\partial x^3}\,v(t,x+w)\Big|^2 dw\\
&\leq h\int_0^h\Big|\frac{\partial^3}{\partial x^3}\,v(t,x+w)\Big|^2 dw\\
&= h\int_x^{x+h}\Big|\frac{\partial^3}{\partial z^3}\,v(t,z)\Big|^2 dz,
\end{aligned}
$$

and, following the same steps as in the proof of (1), we finally obtain

$$
\sum_{x\in Z_h} r^2(x)\Big|\Big(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\Big)v(t,x)\Big|^2\rho^4(x)h\leq h^2N|v(t)|^2_{W^{3,2}}(r,\rho),
$$

with N a constant independent of h, and (2) is proved. □

Finally, we prove the convergence of the discrete problem's generalized solution to the PDE problem's generalized solution, and compute a convergence rate. The accuracy obtained is of order 1.

The convergence result is obtained by imposing additional regularity over the exact solution of problem (2.2) in order to use Theorem 4, but lesser than it is assumed in the correspondent result in [7], for the multidimensional case (see Remark 4).

Theorem 5. *Assume that the hypotheses of Theorems 2 and 3 are satisfied. Assume additionally that (3) in Assumption 2 holds and that $\rho(x)\geq C$ on $\mathbb{R}$, with $C>0$ a constant. Denote by u the solution of (2.2) in Theorem 2 and by u_h the solution of (3.1) in Theorem 3. Assume also that $u\in L^2([0,T];W^{3,2}(r,\rho))$. Then*

$$
\begin{aligned}
&\sup_{0\leq t\leq T}|u(t)-u_h(t)|^2_{l^{0,2}(r)}+\int_0^T|u(t)-u_h(t)|^2_{l^{1,2}(r,\rho)}\,dt\\
&\leq h^2N\int_0^T|u(t)|^2_{W^{3,2}(r,\rho)}dt+N\Big(|g-g_h|^2_{l^{0,2}(r)}+\int_0^T|f(t)-f_h(t)|^2_{l^{0,2}(r)}\,dt\Big),
\end{aligned}
$$

for some constant N independent of h.

Remark 9. Under the conditions of the above theorem, there are modifications in x such that the data $f(t)$ and g are continuous in x, for every $t\in[0,T]$ (see Remark 7). We will consider these modifications in the theorem's proof.

Proof. (Theorem 5) From (2.2) and (3.1), we see that $u-u_h$ satisfies the problem

$$
\begin{cases}
L_h(u-u_h)-\dfrac{\partial}{\partial t}(u-u_h)+(L-L_h)u+(f-f_h)=0 & \text{in } [0,T]\times Z_h\\
(u-u_h)(0,x)=(g-g_h)(x) & \text{in } Z_h.
\end{cases}
\tag{4.7}
$$

Taking in mind Remark 9, $f - f_h \in L^2([0,T]; l^{0,2}(r))$ and $g - g_h \in l^{0,2}(r)$. In what concerns the term $(L - L_h)u$, note that if $u(t) \in W^{3,2}(r,\rho)$, for all $t \in [0,T]$,

$$\begin{aligned}&\sum_{x\in Z_h} r^2(x)|(L-L_h)u(t)|^2\, h\\ &= \sum_{x\in Z_h} r^2(x)\Big|a(t,x)\Big(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+\Big)u(t,x) + b(t,x)\Big(\frac{\partial}{\partial x} - \partial^+\Big)u(t,x)\Big|^2 h < \infty,\end{aligned}$$

owing to (2) in Assumption 3 and to Theorem 4. Thus $(L-L_h)(t)u(t) \in l^{0,2}(r)$ for every $t \in [0,T]$, and by continuity in t we have $(L-L_h)u \in L^2([0,T]; l^{0,2}(r))$.

We have shown that problem (4.7) satisfies the hypotheses of Theorem 3, therefore holding the estimate

$$\begin{aligned}&\sup_{0\le t\le T} |u(t) - u_h(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}(r,\rho)}\, dt\\ &\le N(|g-g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\, dt + \int_0^T |(L-L_h)u(t)|^2_{l^{0,2}(r)}\, dt).\end{aligned}$$

Owing again to (2) in Assumption 3 and to Theorem 4, the result follows. □

Next result is an immediate consequence of Theorem 5.

Corollary 1. *Assume that the hypotheses of Theorem 5 are satisfied, and denote by u the solution of (2.2) in Theorem 2 and by u_h the solution of (3.1) in Theorem 3. If there is a constant N independent of h such that*

$$|g-g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t)-f_h(t)|^2_{l^{0,2}(r)} dt \le h^2 N\Big(|g|^2_{W^{2,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{1,2}(r,\rho)} dt\Big)$$

then

$$\begin{aligned}&\sup_{0\le t\le T} |u(t)-u_h(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t)-u_h(t)|^2_{l^{1,2}(r,\rho)} dt\\ &\le h^2 N\Big(\int_0^T |u(t)|^2_{W^{3,2}(r,\rho)} dt + |g|^2_{W^{2,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{1,2}(r,\rho)} dt\Big).\end{aligned}$$

5. Final remarks

By considering a suitable class of weighted Sobolev spaces, and its zero and first-order discrete versions, we could deal with the growth of the PDE coefficients with respect to the space variable, under a weak setting, and prove stability, consistency and convergence properties of the numerical scheme.

Moreover, these results were obtained under regularity assumptions weaker than those required in the corresponding consistency and convergence results in [7], for the more general case of multidimensional PDEs.

Possible further research directions include: the approach of the degenerate case (for this, the results obtained in [10, 11] will play a central role); the use of splitting-up methods (see [9]), following Richardson's idea to accelerate numerical schemes.

References

[1] Barles, G., Ch. Daher, and M. Romano, 1995: Convergence of numerical schemes for parabolic equations arising in Finance theory. *Math. Models Methods Appl. Sci.*, 5(1), 125–143.

[2] Boyle P. P., and Y. Tian, 1998: An explicit finite difference approach to the pricing of barrier options. *Appl. Math. Finance*, 5, 17–43.

[3] Broadie, M., and J. B. Detemple, 2004: Option pricing: valuation models and applications. *Manag. Sci.*, 50(9), 1145–1177.

[4] Evans, L. C., 1998: *Partial Differential Equations.* Graduate Studies in Mathematics 19, American Mathematical Society, U.S.A.

[5] Gilli, M., E. Kellezi, and G. Pauletto, 2002: Solving finite difference schemes arising in trivariate option pricing. *J. Econ. Dynam. Control*, 26(9-10), 1499–1515.

[6] Gonçalves, F. F., 2007: Numerical approximation of partial differential equations arising in financial option pricing. Ph.D. Thesis. University of Edinburgh, U.K. Eprint: http://cemapre.iseg.utl.pt/archive/preprints/thfernando.pdf.

[7] Gonçalves, F. F., and M. R. Grossinho, 2009: Numerical approximation of Cauchy problems for multidimensional PDEs with unbounded coefficients arising in Financial Mathematics. Eprint: http://cemapre.iseg.utl.pt/archive/preprints/09GonGrosWSobolev.pdf.

[8] Gyöngy, I., and N. V. Krylov, 1990: Stochastic partial differential equations with unbounded coefficients and applications I. *Stochastics*, 32, 53–91.

[9] Gyöngy, I., and N. V. Krylov, 2005: An accelerated splitting-up method for parabolic equations. *SIAM J. Math. Anal.*, 37, 1070–1097.

[10] Gyöngy, I., and N. V. Krylov, 2008a: First derivatives estimates for finite-difference schemes. *Math. Comp*, American Mathematical Society (electronic).

[11] Gyöngy, I., and N. V. Krylov, 2008b: Higher order derivative estimates for finite-difference schemes. Eprint: http://arxiv.org/abs/0805.3149.

[12] Lamberton, D., and B. Lapeyre, 1996: *Introduction to Stochastic Calculus Applied to Finance*. Chapman and Hall, U.K.

[13] Lions, J.-L., 1969: *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires* (in French). Dunod, Gauthier-Villars, Paris, France .

[14] Lötstedt, P., J. Persson, L. von Sydow, and J. Tysk, 2007: Spacetime adaptive finite difference method for European multi-asset options. *J. Comput. Math. Appl.*, 53(8), 1159–1180.

[15] Purtukhia, O. G., 1984: On the equations of filtering of multi-dimensional diffusion processes (unbounded coefficients) (in Russian). Thesis. Lomonosov University, Moscow.

[16] Purtukhia, O. G., 1984: Innovation problem for degenerate processes (unbounded coefficients) (in Russian). *Uspekhi Math. Nauk.*, 39(4).

[17] Purtukhia, O. G., 1984: On the representation of the solution of the Cauchy problem. *Abstracts of the XVIII School-Seminar in Probability Theory and Statistics*, 40, Bakhuriani.

[18] Purtukhia, O. G., 1984: On the Cauchy problem for linear stochastic partial differential equations (unbounded coefficients). *Stochastic Analysis and Assymptotic Problems in Probability Theory and Statistics*, 57-71.

[19] Tangman D. Y., A. Gopaul, and M. Bhuruth, 2008: Numerical pricing of options using high-order compact finite difference schemes. *J. Comput. Appl. Math.*, 218, 270–280.

[20] Thomée, V., 1990: Finite Difference Methods for Linear Parabolic Equations. *Finite Difference Methods (Part 1) – Solution of Equations in* $\mathbb{R}^n$ *(Part 1)*, P. G. Ciarlet, and J. L. Lions, Eds., Handbook of Numerical Analysis, Vol. I, North-Holland, 3–196.

CEMAPRE, ISEG, Technical University of Lisbon, Rua do Quelhas 6, 1200-781 Lisboa, Portugal, Also with CMA, Mathematical Sciences Institute, Bldg 27, The Australian National University, Canberra, ACT 0200, Australia

E-mail address: fgoncalves@iseg.utl.pt, fernando.goncalves@anu.edu.au

CEMAPRE, ISEG, Technical University of Lisbon, Rua do Quelhas 6, 1200-781 Lisboa, Portugal

E-mail address: mrg@iseg.utl.pt