Cramér-Lundberg results for the infinite time ruin probability in the compound binomial model

Bjørn Sundt*†
Storebrand Life Insurance, Oslo, Norway

Alfredo D. Egídio dos Reis*
CEMAPRE and ISEG, Technical University of Lisbon, Portugal

December 27, 2006

1 Introduction

The compound binomial model for the risk process was introduced by Gerber (1988) and is sometimes considered as a discrete time approximation to the classical compound Poisson model in continuous time; Dickson (1994) discusses this issue.

After having introduced some notation in Section 2, we describe the model and set up some recursions for the infinite time ruin probability in Section 3.

The core of the paper is Section 4. Here we present the Lundberg inequality and the Cramér-Lundberg approximation for the infinite time ruin probability in the compound binomial model and characterise the class of severity distributions for which the asymptotic expression is exact.

Finally, in Section 5, we compare this characterisation with the analogous characterisation in the continuous time Poisson model. Although it is well-known in the latter model, we give a deduction comparable with the one in Section 4.

*Support from *Fundação para Ciência e Tecnologia* - FCT/POCTI is gratefully acknowledged.

†A part of the present research was carried out while the first author stayed at CEMAPRE/ISEG in Lisbon October 2004.

2 Notation

We denote a cumulative distribution function (briefly referred to as distribution) by a capital letter, its tail by that letter with superscript c, and its mean, moment generating function, and probability generating function by that letter as subscript to μ, φ, and τ. Its probability function or density is denoted by the corresponding lower case letter. Thus, f would be the probability function or density of a distribution F, and this distribution has tail F^{c}, mean μ_F, moment generating function φ_F, and probability generating function τ_F, to the extent that these quantities exist.

By the notation

$$a(x) \sim b(x), \qquad (x \uparrow \infty)$$

we shall mean that $\lim_{x\uparrow\infty} a(x)/b(x) = 1$.

3 The model

3A. We consider an insurance portfolio over time. The units of time and money are chosen such that the premium for each time unit is equal to one. Let X_i denote the aggregate claims of time unit i. We assume that the X_is are non-negative, integer-valued, independent, and identically distributed with distribution G with

$$\mu_G < 1, \tag{1}$$

that is, the premium has a positive risk loading.

For $n = 0, 1, 2, \ldots$, we let $S_n = \sum_{i=1}^{n} X_i$ be the aggregate claims up to time n and U_n the reserve at time n. Denoting the initial reserve at time zero by u, we have $U_n = u + n - S_n$, that is,

$$U_0 = u$$
$$U_n = U_{n-1} + 1 - X_n. \qquad (n = 1, 2, 3, \ldots)$$

We define the infinite time probability of ruin $\psi(u)$ as the probability that the reserve becomes strictly negative at some time, that is,

$$\psi(u) = \Pr\left(\cup_{n=1}^{\infty} (U_n < 0)\right). \qquad (u = 0, 1, 2, \ldots)$$

By conditioning on the aggregate claim amount of the first time unit, we immediately obtain

$$\psi(u) = G^{\mathrm{c}}(u+1) + \sum_{x=0}^{u+1} g(x)\,\psi(u+1-x), \qquad (u = 0, 1, 2, \ldots) \tag{2}$$

which solved with respect to $\psi(u+1)$ can be used for recursive evaluation of ψ.

3B. The following parameterisation of this model is often used in the literature: Let

$$q = \Pr(X_i = 0); \quad p = \Pr(X_i > 0) = 1 - q$$

$$F(x) = \Pr(X \le x \,|X > 0) = \frac{G(x) - G(0)}{p} \qquad (x = 1, 2, 3, \dots)$$

$$f(x) = \Pr(X = x \,|X > 0) = \frac{g(x)}{p}. \qquad (x = 1, 2, 3, \dots)$$

In the present parameterisation, we can rewrite (2) as

$$\psi(u) = q\psi(u+1) + p\left(F^{\mathrm{c}}(u+1) + \sum_{x=1}^{u+1} f(x)\,\psi(u+1-x)\right). \qquad (3)$$

$$(u = 0, 1, 2, \dots)$$

When the present parameterisation is used, one would normally let the time unit be so short that it could be assumed that at most one claim can occur per time unit. Then p becomes the probability that a claim occurs, and F the distribution of the severity of that claim.

4 Cramér-Lundberg results

4A. We assume that there exists an $R > 0$ such that

$$\mathrm{E}\, e^{R(X_i - 1)} = 1. \qquad (4)$$

Then it follows from e.g. Lemma 10.1 in Sundt (1999) that $\psi(u) \le e^{-Ru}$ for $u = 0, 1, 2, \dots$. This is the Lundberg inequality.

We see that (4) can be written as $\varphi_G(R) = e^R$. However, as we now work in a discrete framework, it will be more convenient to work with probability generating functions than with moment generating functions, so we assume that there exists a $w > 1$ such that

$$\tau_G(w) = w. \qquad (5)$$

Then the Lundberg inequality can be written as

$$\psi(u) \le w^{-u}. \qquad (u = 0, 1, 2, \dots) \qquad (6)$$

In terms of the other parameterisation, we can write the Lundberg condition (5) as

$$q + p\tau_F(w) = w. \tag{7}$$

Shiu (1989) shows that

$$\psi(0) = \frac{p}{q}(\mu_F - 1). \tag{8}$$

Liu et al. (2005) present a generalisation of the compound binomial model, and within their model, they deduce an asymptotic result for $\psi(u)$. As a special case, this result gives that if (7) is satisfied in our model, then

$$\psi(u) \sim \frac{1 - p\mu_F}{p\tau_F'(w) - 1} w^{-(u+1)} = \frac{1 - \mu_G}{\tau_G'(w) - 1} w^{-(u+1)}; \qquad (u \uparrow \infty) \tag{9}$$

see also Cosette et al. (2004). This is a Cramér-Lundberg approximation for the ruin probability.

4B. Let us now assume that the Cramér-Lundberg approximation (9) holds exact, that is, that there exists a k such that

$$\psi(u) = kw^{-u}. \qquad (u = 0, 1, 2, \dots) \tag{10}$$

We want to characterise the class of severity distributions F for which this result holds.

As $\psi(0) = k$, we must have $k \leq 1$. Furthermore, if $k = 1$, then (3) gives that $\psi(u) = 1$ for all positive integers u. This contradicts the assumption that $w > 1$, so we must have $k < 1$.

Theorem 1. *If* (10) *should hold, then*

$$f(x) = \begin{cases} \rho & (x = 1) \\ (1-\rho)(1-\alpha)\alpha^{x-2} & (x = 2, 3, 4, \dots) \end{cases} \tag{11}$$

with

$$\rho = \frac{pw - k(w - q)}{pw(1-k)}; \quad \alpha = \frac{1 - kw}{w(1-k)}. \tag{12}$$

Proof. By replacing u with $u-1$ in (3), we obtain that for $u = 1, 2, 3, \dots$

$$\psi(u-1) = q\psi(u) + p\left(F^{c}(u) + \sum_{x=1}^{u} f(x)\psi(u-x)\right).$$

Insertion of (10) and multiplication by w^u gives

$$kw = kq + p\left(F^{c}(u)\,w^{u} + k\sum_{x=1}^{u} f(x)\,w^{x}\right). \qquad (13)$$

In particular, for $u = 1$, we obtain

$$kw = kq + p\left((1 - f(1))\,w + kf(1)\,w\right),$$

which gives

$$f(1) = \frac{pw - k(w - q)}{pw(1 - k)} = \rho. \qquad (14)$$

By replacing u with $u - 1$ in (13), we obtain

$$kw = kq + p\left(F^{c}(u-1)\,w^{u-1} + k\sum_{x=1}^{u-1} f(x)\,w^{x}\right), \qquad (u = 2, 3, 4, \dots)$$

and subtraction of this from (13) and division by pw^{u-1} gives

$$F^{c}(u)\,w - F^{c}(u-1) + kf(u)\,w = 0. \qquad (u = 2, 3, 4, \dots) \qquad (15)$$

By replacing u with $u - 1$, we obtain

$$F^{c}(u-1)\,w - F^{c}(u-2) + kf(u-1)\,w = 0. \qquad (u = 3, 4, 5, \dots)$$

and subtraction of (15) gives

$$wf(u) - f(u-1) + kw\left(f(u-1) - f(u)\right) = 0, \qquad (u = 3, 4, 5, \dots)$$

from which we obtain

$$f(u) = \frac{1 - kw}{w(1 - k)} f(u-1) = \alpha f(u-1), \qquad (u = 3, 4, 5, \dots)$$

so that

$$f(u) = f(2)\,\alpha^{u-2}. \qquad (u = 2, 3, 4, \dots) \qquad (16)$$

We have

$$w(1 - k) - (1 - kw) = w - 1 > 0$$

so that

$$\alpha = \frac{1 - kw}{w(1 - k)} < 1.$$

As F is a distribution, application of (14) and (16) gives

$$1 = \sum_{x=1}^{\infty} f(x) = \rho + f(2) \sum_{x=2}^{\infty} \alpha^{u-2} = \rho + \frac{f(2)}{1-\alpha},$$

so that $f(2) = (1-\rho)(1-\alpha)$, which together with (14) and (16) proves Theorem 1. Q.E.D.

The following corollary follows easily from Theorem 1 by solving (12) for k and w and insertion in (10).

Corollary 1. *If f satisfies* (11), *then*

$$\psi(u) = \frac{p(1-\rho)}{q(1-\alpha)} \left(\frac{p}{q}(1-\rho) + \alpha \right)^u . \qquad (u = 0, 1, 2, \dots) \tag{17}$$

We easily obtain that in the present case we have

$$\mu_F = \rho + (1-\rho)\frac{2-\alpha}{1-\alpha} = 1 + \frac{1-\rho}{1-\alpha}. \tag{18}$$

Insertion in (17) gives

$$\psi(u) = \frac{p}{q}(\mu_F - 1) \left(\frac{p}{q}(1-\rho) + \alpha \right)^u , \qquad (u = 0, 1, 2, \dots)$$

which is consistent with (8).

Let us look at the condition (1). From (18), we obtain that in the present case we have

$$\mu_G = p\left(1 + \frac{1-\rho}{1-\alpha}\right).$$

Hence, (1) gives that

$$p\left(1 + \frac{1-\rho}{1-\alpha}\right) < 1.$$

which we rewrite in the following two ways:

$$\frac{p(1-\rho)}{q(1-\alpha)} < 1; \qquad \frac{p}{q}(1-\rho) + \alpha < 1.$$

The first inequality shows that the constant factor in (17) is less than one, and the second that the power part is less than one. In the limiting case $\mu_G = 1$, both these expressions approach one, so that the ruin probability goes to one, which is reasonable.

Let us now look at some special cases of Corollary 1:

1. $\rho = 1$. In this case, F is concentrated in one so that all claims are equal to one. Hence, the claims will never exceed the premiums, so that the reserve will never decrease. Thus, $\psi(u) = 0$ for $u = 0, 1, 2, \ldots$.

2. $\alpha = 0$. In this case, the claims cannot exceed two. We obtain

   $$\psi(u) = \left(\frac{p}{q}(1-\rho)\right)^{u+1}, \qquad (u = 0, 1, 2, \ldots)$$

   which is given by Willmot (1993).

   In the special case $\rho = 0$, the claims are concentrated in two. This corresponds to the gambler's ruin problem discussed by Shiu (1989), Willmot (1993), and Sundt (1999) (roulette example in Chapter 10). We obtain that $\psi(u) = (p/q)^{u+1}$ for $u = 0, 1, 2, \ldots$.

3. $\rho = 1 - \alpha$. This is the shifted geometric distribution given by $f(x) = (1-\alpha)\alpha^{x-1}$ for $x = 1, 2, 3, \ldots$. We obtain

   $$\psi(u) = \frac{p}{1-\alpha}\left(\frac{\alpha}{q}\right)^{u+1}. \qquad (u = 0, 1, 2, \ldots)$$

   This case is considered by Willmot (1993) and Dickson (1994).

4. $\rho = 0$. This is the shifted geometric distribution given by $f(x) = (1-\alpha)\alpha^{x-2}$ for $x = 2, 3, 4, \ldots$. We obtain

   $$\psi(u) = \frac{p}{q(1-\alpha)}\left(\frac{p}{q} + \alpha\right)^{u}. \qquad (u = 0, 1, 2, \ldots)$$

The cases 2 and 3 are treated in a more general compound Markov binomial model by Cossette et al. (2004).

5 Comparison with the continuous time compound Poisson model

5A. The compound binomial model is sometimes used as an approximation to the continuous time compound Poisson model. In the present section, we shall present the equivalent results for the latter model to results shown for the former model in the previous section. We do not present the compound Poisson model in its most general form, but, as the purpose of the presentation is just to indicate the relation to the binomial model, we make the simplifying assumption that the claim amounts are continuously distributed.

We assume that claims occur in continuous time with intensity λ independent of the occurrence times and amounts of other claims. The claim amounts are independent and identically distributed on the positive numbers with continuous distribution F. The premium is paid continuously, and the units of time and money are chosen such that the rate is equal to one. It is assumed that the premium has a positive risk loading, that is,

$$\lambda \mu_F < 1. \tag{19}$$

Let S_t denote the aggregate claims up to time t and U_t the reserve at time t. Denoting the initial reserve at time zero by u, we obtain that $U_t = u+t-S_t$ for $t \geq 0$. Like in the compound binomial model, we define the infinite time ruin probability $\psi(u)$ as the probability that the reserve becomes strictly negative at some time, that is,

$$\psi(u) = \Pr\left(\cup_{t>0}\left(U_t < 0\right)\right). \qquad (u \geq 0)$$

We shall now deduce an integro-differential equation corresponding to the recursion (3). By infinitesimal reasoning, conditioning on what happens in the time interval $(0, h)$ for some small $h > 0$, we immediately obtain

$$\psi(u) = (1-\lambda h)\,\psi(u+h) + \lambda h\left(F^{\mathrm{c}}(u+h) + \int_0^{u+h} \psi(u+h-x)\, f(x)\, dx\right),$$

which we rewrite as

$$\frac{\psi(u+h)-\psi(u)}{h} = \lambda\left(\psi(u+h) - F^{\mathrm{c}}(u+h) - \int_0^{u+h} \psi(u+h-x)\, f(x)\, dx\right).$$

By letting $h \downarrow 0$, we obtain

$$\frac{d^+}{du}\psi(u) = \lambda\left(\psi(u) - F^{\mathrm{c}}(u) - \int_0^{u} \psi(u-x)\, f(x)\, dx\right). \tag{20}$$

5B. We assume that there exists an $R > 0$ such that

$$\varphi_F(R) = \frac{R}{\lambda} + 1. \tag{21}$$

Then it is well known that $\psi(u) \leq e^{-Ru}$ for $u \geq 0$; cf. e.g. Section 13.4 in Bowers et al. (1997) or Section 10.3 in Sundt (1999).

Analogous to (8) and (9), we have

$$\psi(0) = \lambda\mu_F \tag{22}$$

$$\psi(u) \sim \frac{1-\lambda\mu_F}{\lambda\varphi_F'(R)-1}e^{-Ru}; \qquad (u\uparrow\infty) \tag{23}$$

cf. e.g. Example (a) in Section XI.7 in Feller (1971).

5C. Let us now assume that the Cramér-Lundberg approximation (23) holds exact, that is, there exists a $k \leq 1$ such that

$$\psi(u) = ke^{-Ru}. \qquad (u \geq 0) \tag{24}$$

We want to characterise the class of severity distributions F for which this result holds.

Theorem 2. *If* (24) *should hold, then we must have*

$$k = \frac{\lambda}{\lambda+R} \tag{25}$$

$$f(x) = \xi e^{-\xi x} \qquad (x > 0) \tag{26}$$

with

$$\xi = \frac{R}{1-k}. \tag{27}$$

Proof. Insertion of (24) in (20) and multiplication with e^{Ru} gives

$$-kR = \lambda\left(k - F^{c}(u)e^{Ru} - k\int_0^u e^{Rx}f(x)\,dx\right),$$

from which we obtain

$$F^{c}(u)e^{Ru} = k\left(\frac{R}{\lambda}+1-\int_0^u e^{Rx}f(x)\,dx\right). \tag{28}$$

In particular, for $u=0$, this gives

$$1 = k\left(\frac{R}{\lambda}+1\right),$$

from which we obtain (25).

Differentiation of (28) with respect to u gives

$$-f(u)e^{Ru} + F^{c}(u)Re^{Ru} = -ke^{Ru}f(u),$$

from which we obtain

$$\frac{f(u)}{F^{c}(u)} = \frac{R}{1-k} = \xi.$$

Thus, F has constant failure rate ξ and must then be the exponential distribution given by (26). Q.E.D.

The following corollary follows easily from Theorem 2 by solving (25) and (27) for k and R and insertion in (24).

Corollary 2. *If f satisfies* (26)*, then*

$$\psi(u) = \frac{\lambda}{\xi} e^{-(\xi-\lambda)u}. \qquad (u \geq 0) \tag{29}$$

As now $\mu_F = 1/\xi$, we can rewrite (29) as

$$\psi(u) = \lambda \mu_F e^{-(\xi-\lambda)u}, \qquad (u \geq 0)$$

which is consistent with (22).

The condition (19) can be written in the following two ways:

$$\lambda/\xi < 1; \qquad \xi - \lambda > 0.$$

Like in the discrete case, these two inequalities ensure that each of the two factors in (29) is less than one, and in the limiting case $\lambda\mu_F = 1$, both of them and the ruin probability go to one.

In Example (b) in Section XIV.2 of Feller (1971), Theorem 2 is proved by using Laplace transforms. We have used the present proof for easier comparison with our proof of Theorem 1. We also refer to Section 13.6 in Bowers et al. (1997).

It is interesting to compare Theorems 1 and 2. As pointed out earlier, the discrete compound binomial model is sometimes used as an approximation to the continuous compound Poisson model. In the latter model, the Cramér-Lundberg approximation holds exact only when the severity distribution is exponential. The geometric distribution is the discrete analogue to the exponential distribution, and it is then natural that in the discrete case, the Cramér-Lundberg approximation will hold exact for this severity distribution. However, in the discrete case, it is exact for a wider class, allowing for severity distributions with a limited range.

References

BOWERS, N.L. et al. (1997). *Actuarial Mathematics.* (2. ed.). Society of Actuaries, Itasca, Illinois.

COSSETTE, H. et al. (2004). Exact expressions and upper bounds for ruin probabilities in the compound Markov binomial model. *Insurance: Mathematics and Economics* **24**, 449–466.

DICKSON, D.C.M. (1994). Some comments on the compound binomial model. *ASTIN Bulletin* 24, 33–44.

FELLER, W. (1971). *An introduction to probability theory and its applications.* Vol. 2 (2. ed.). Wiley, New York.

GERBER, H.U. (1988). Mathematical fun with the compound binomial model. *ASTIN Bulletin* **18**, 161–168.

LIU, G. et al. (2005). Ruin probability in the continuous time compound binomial model. *Insurance: Mathematics and Economics* **36**, 303–316.

SHIU, E. (1989). The probability of eventual ruin in the compound binomial model. *ASTIN Bulletin* **19**, 179–190.

SUNDT, B. (1999). *An introduction to non-life insurance mathematics.* (4. ed.) Verlag Versicherungswirtschaft GmbH, Karlsruhe.

WILLMOT, G.E. (1993). Ruin probabilities in the compound binomial models. *Insurance: Mathematics and Economics* **12**, 132–142.

Bjørn Sundt
Storebrand Life Insurance
P.O. Box 1380 Vika
N–0114 Oslo
NORWAY

Alfredo D. Egídio dos Reis
Departmento de Matemática
ISEG, Universidade Técnica de Lisboa
Rua do Quelhas 2
1200–781 Lisboa
PORTUGAL

Abstract

In the present paper, we characterise the class of severity distributions for which the Cramér-Lundberg approximation for the infinite time ruin probability in the compound binomial model is exact, and we compare this characterisation with the continuous time classical compound Poisson model.