On a Sparre-Andersen risk model with PH(n) interclaim times

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Abstract

For actuarial applications we consider the Sparre–Andersen risk model when the interclaim times follow a Phase–Type distribution, PH(n).

First of all we focus our attention on the generalized Lundberg's equation to determine the cases when multiple roots can arise, with the highest possible level of accuracy. Second, we study the linear independence of the eigenvectors related to the Lundberg's matrix . Finally we apply our results to compute the ultimate and finite time ruin probabilities, the probability of arrival to a barrier prior to ruin, severity of ruin and its maximum, the expected discounted future dividends, among others.

Keywords: Sparre–Andersen risk model; generalized Lundberg's equation; Phase-Type (n) interclaim times; linearly independent eigenvectors; maximum severity of ruin; probability of reaching an upper barrier; maximum severity of ruin, discounted dividends.

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1 Introduction

Lundberg's equation, named after Swedish actuary Ernst Filip Oskar Lundberg, is a major subject of study for the computation of ruin probabilitites. This equation came first into light for the study for the well known Lundberg's inequality and adjustment coefficient, being the former as an upper bound for the ultimate ruin probability. Nowadays, the roots of the Lundberg's equation play an important role in the calculation of many quantities that are fundamental in risk and ruin theory. Namely, the ultimate and finite time ruin probabilities, the probability of arrival to a barrier prior to ruin, severity of ruin and its maximum, the expected discounted future dividends, among others.

All those calculations depend on the nature of the roots of the Lundberg's equation, particularly its roots with positive real parts. There are several papers that have been devoted to the subject, namely Albrecher and Boxma (2005), Dickson and Waters (2004), Li and Garrido (2004a,b), Ren (2007) and some others. But in all those works it is always assumed that the roots are distinct.

Our interest is to address two problems: First, to determine the cases when multiple roots can arise, with the highest possible level of accuracy; Second, to study the linear independence of the eigenvectors related to the Lundberg's matrix. We will then be able to compute the quantities discussed above for the case of multiple roots.

We illustrate finding explicit formulae for some examples and values for the parameter n of the PH(n) family, and some particular claim amount distributions.

2 Phase–Type model

In the present article we work with the Sparre–Andersen model driven by the equation

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0,$$

where u (≥ 0) is the initial capital, c (≥ 0) is the premium income per unit time t, $\{X_i\}_{i=1}^{\infty}$ is a sequence of (i.i.d.) independent and identically distributed random variables, each representing a single claim amount, with common distribution function $P(x)$ and density $p(x)$. Its Laplace transform is denoted by $\hat{p}(.)$. Denote by $\mu_k = E[X_1^k]$ the k-th moment of X_i. We assume the existence of μ_1(general condition), in some parts of this manuscript we will work with cases where higher moments exist. The sequence $\{X_i\}$ is independent of the counting process $\{N(t), t \geq 0\}$, with $N(t) = \max\{k : W_1 + W_2 + \cdots + W_k \leq t\}$ where the random variables W_i, $i \in \mathbb{N}^+$, are i.i.d. with cumulative distribution $K(t)$ and density $k(t)$. The Laplace transform is denoted by $\hat{k}(.)$

We assume that the interclaim times W_i follow a Phase–Type(n) distribution with representation $(\alpha, \mathbf{B})$. This means that W_i corresponds to the time of absorption in a terminating continuous time Markov chain $\{J(t)\}_{t\geq 0}$ with n transient states $\{1, 2, \ldots, n\}$ and one absorbing state $\{0\}$. The $n \times n$ intensity matrix $\mathbf{B} = (b_{i,j})_{i,j=1}^{n}$ denotes the transition rates between the n transient states, with $b_{i,i} < 0$, $b_{i,j} \geq 0$ for $i \neq j$, and $\sum_{j=1}^{n} b_{i,j} \leq 0$ for $i = 1, \ldots, n$. The vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ denotes the initial distribution with $\alpha_i \geq 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^{n} \alpha_i = 1$. Then

$$\begin{aligned} k(t) &= \alpha e^{\mathbf{B}t}\mathbf{b}^{\mathsf{T}}, \quad K(t) = 1 - \alpha e^{\mathbf{B}t}\mathbf{e}^{\mathsf{T}}, \quad t \geq 0, \\ \hat{k}(s) &= \alpha(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}}, \quad E[W_1] = -\alpha\mathbf{B}\mathbf{e}^{\mathsf{T}}, \end{aligned}$$

where $\mathbf{b}^\intercal = -\mathbf{B}\mathbf{e}^\intercal$ is the vector of exit rates to the absorbing state $\{0\}$, $\mathbf{e} = (1, 1, \ldots, 1)$ is a $1 \times n$ vector and $\mathbf{I}$ is the $n \times n$ identity matrix.

We assume a positive loading factor, that is $cE[W_1] > E[X_1]$.

3 Lundberg's equation

The following matrix

$$\mathbf{L}_\delta(s) = \left(s - \frac{\delta}{c}\right)\mathbf{I} + \frac{1}{c}\mathbf{B} + \frac{1}{c}\mathbf{b}^\intercal \alpha \hat{p}(s), \tag{3.1}$$

which we call the Lundberg's matrix, have been subject of study in several works, like Albrecher and Boxma (2005), Ren (2007), Li (2008), Ji (2011), among others. In the expression δ stands for a non negative constant.

According to Ren (2007), the solutions of

$$Det(\mathbf{L}_\delta(s)) = 0, \tag{3.2}$$

and the solutions of the fundamental Lundberg's equation

$$\hat{k}(\delta - cs)\hat{p}(s) = 1, \tag{3.3}$$

as defined in Gerber and Shiu (2005) are identical.

Albrecher and Boxma (2005) show that (3.2) has exactly n solutions in the right half of the complex plane using matrix theory, therefore the fundamental Lundberg's equation (3.3) have exactly the same n solutions in the right half of the complex plane, which we denote by $\rho_1, \rho_2, \ldots, \rho_n$.

In all the papers mentioned before, it is assumed that these roots have distinct values. However, we can find a great variety of examples where multiple roots can arise, specially double roots. First of all we want to show how to build these examples with double roots.

Definition 3.1. *Let* $\mathbf{A} = (a_{i,j})_{i,j=1}^n$ *be a* $n \times n$ *matrix.*

Define, for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$

$$\mathbf{M}_{i_1,i_2\ldots i_k} = \begin{pmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \ldots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \ldots & a_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \ldots & a_{i_k,i_k} \end{pmatrix},\ 1 \leq k \leq n,$$

then

$$tr_k(\mathbf{A}) = \sum_{1\leq i_1<i_2<\ldots<i_k\leq n} det(\mathbf{M}_{i_1,i_2\ldots i_k}).$$

Example 3.1. *For* $k = 1$

$$tr(\mathbf{A}) = \sum_{i=1}^n \mathbf{M}_i = \sum_{i=1}^n a_{ii}.$$

For $k = 2$

$$tr_2(\mathbf{A}) = \sum_{1\leq i<j\leq n} det(\mathbf{M}_{ij}) = \sum_{1\leq i<j\leq n} (a_{ii}a_{jj} - a_{ij}a_{ji}).$$

For $k = n - 1$

$$tr_{n-1}(\mathbf{A}) = \sum_{i=1^n} det(\mathbf{M}_{1\ldots,i-1,i+1,\ldots,n}) = det(\mathbf{A})tr(\mathbf{A}^{-1}).$$

For $k = n$

$$tr_n(\mathbf{A}) = det(\mathbf{M}_{1\ldots n}) = det(\mathbf{A}).$$

By convention we set $tr_0(A) = 1$*.*

Theorem 3.2.

$$\hat{k}(s) = \alpha(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}} = \frac{N(s,n)}{det(s\mathbf{I} - \mathbf{B})},$$

where,

$$det(s\mathbf{I} - \mathbf{B}) = \sum_{i=0}^{n} (-1)^{n-i} tr_{n-i}(\mathbf{B}) s^i,$$

and, for n *odd*

$$\begin{aligned}
N(s,n) \;\; = \;\; & \alpha[-\mathbf{B}(s)^{n-1} - [\mathbf{B}^2 - \mathbf{B}tr(\mathbf{B})](s)^{n-2} - \ldots + [(-1)^{\frac{n-1}{2}}\mathbf{B}^{\frac{n-1}{2}} \\
& -(-1)^{\frac{n-3}{2}}\mathbf{B}^{\frac{n-3}{2}}tr(\mathbf{B}) - \ldots - \mathbf{B}tr_{\frac{n-3}{2}}(\mathbf{B})](s)^{\frac{n+1}{2}} \\
& -[(-1)^{\frac{n+1}{2}}\mathbf{B}^{\frac{n+1}{2}} - (-1)^{\frac{n-1}{2}}\mathbf{B}^{\frac{n-1}{2}}tr(\mathbf{B}) - \ldots \\
& -\mathbf{B}tr_{\frac{n-1}{2}}(\mathbf{B})](s)^{\frac{n-1}{2}} - [(-1)^{\frac{n-1}{2}}\mathbf{B}^{-\frac{n-3}{2}}det(\mathbf{B}) \\
& -(-1)^{\frac{n-3}{2}}\mathbf{B}^{-\frac{n-5}{2}}tr_{n-1}(\mathbf{B}) - \ldots - \mathbf{I}tr_{\frac{n+3}{2}}(\mathbf{B})](s)^{\frac{n-3}{2}} \\
& - \ldots + [\mathbf{B}^{-1}det(\mathbf{B}) - \mathbf{I}tr_{n-1}(\mathbf{B})](s) - det(\mathbf{B})]\mathbf{1}^{\mathsf{T}},
\end{aligned}$$

and an analogous formula for n *even.*

Example 3.2. *For* $n = 1$*,* $\alpha = (1)$*,* $\mathbf{B} = (-b)$*,* $\mathbf{1} = (1)$*, then*

$$\hat{k}(s) = \frac{\alpha[-\mathbf{B}]\mathbf{1}^{\mathsf{T}}}{s - det(\mathbf{B})} = \frac{b}{s+b}.$$

For $n = 2$

$$\hat{k}(s) = \frac{\alpha[-\mathbf{B}s + \mathbf{I}det(\mathbf{B})]\mathbf{1}^{\mathsf{T}}}{s^2 - tr(\mathbf{B})s + det(\mathbf{B})}.$$

For $n = 3$

$$\hat{k}(s) = \frac{\alpha[-\mathbf{B}s^2 - (\mathbf{B}^2 - \mathbf{B}tr(\mathbf{B}))s - \mathbf{I}det(\mathbf{B})]\mathbf{1}^{\mathsf{T}}}{s^3 - tr(\mathbf{B})s^2 + tr_2(\mathbf{B})s - det(\mathbf{B})}.$$

Now we recall the fundamental Lundberg's equation $\hat{k}(\delta - cs)\hat{p}(s) = 1$. We restrict our attention to the right half of the complex plane, more specifically on the positive real axis, and we look for the possibility of having a double real root.

For $s \in \mathbb{R}^+$, the Laplace transform $\hat{p}(s)$ is a positive and decreasing function of s, with $p(0) = 1$ and $lim_{s \to \infty} \hat{p}(s) = 0$. Therefore $\hat{p}(s)$ has no zeros or poles in $s \in \mathbb{R}^+$.

The function $\hat{k}(\delta - cs)$ is the quotient of the polynomial $N(s, n)$, which has degree at most $n - 1$, and the polynomial $det(s\mathbf{I} - \mathbf{B})$, which has degree n. The poles of $\hat{k}(\delta - cs)$ are the numbers $s = \frac{\delta - \zeta}{c}$, where ζ ranges over all the eigenvalues of $\mathbf{B}$.

Theorem 3.3. *Let s_1 and s_2, with $s_1 < s_2$, be two real poles of $\hat{k}(\delta - cs)$, and suppose that there is no other real pole or zero of $\hat{k}(\delta - cs)$ in the interval (s_1, s_2). If $\hat{k}(\delta - cs)$ is positive in the interval (s_1, s_2) then the fundamental Lundbeg's equation has one of the following:*

- Two real roots in the interval.

- A double root in the interval.

- Two complex conjugate roots, where the real part of them is in the interval.

Example 3.3. *Suppose that the interclaim times W_i follow a generalized Erlang(3) distribution, with intensity matrix*

$$\mathbf{B} = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0 & -1.5 & 1.5 \\ 0 & 0 & -2.5 \end{pmatrix}$$

and $\alpha = (1, 0, 0)$, $\mathbf{b} = (0, 0, 2.5)$. Then $E[W_i] = 3.067$. Suppose that the claim amounts x_i are Exponentially distributed with parameter $\beta \geq 0.5$. Then we choose $c = 1$ to satisfy the positive loading condition and let $\delta = 0.5$.

The fundamental Lundberg's equation becomes

$$\left(\frac{1.875}{(1-s)(2-s)(3-s)}\right)\left(\frac{\beta}{\beta+s}\right)=1$$

The function $\hat{k}(\delta - cs) = \hat{k}(0.5 - s) = \frac{1.875}{(1-s)(2-s)(3-s)}$ *has no zeros and 3 poles at* $s = 1, 2, 3$*, furthermore it is positive in the interval* $(2, 3)$*. Then it is easy to verify that the fundamental Lundberg's equation has*

- Two real roots in $(2, 3)$ *for* $0.5 \leq \beta < 0.67$*.*

- A double root 2.61 *in* $(2, 3)$ *for* $\beta = 0.67$*.*

- Two complex conjugate roots, where the real part of them is in $(2, 3)$ *for* $\beta > 0.67$*.*

4 Lundberg's matrix

Previously we told that the solutions of the fundamental Lundberg's equation and the solutions of (3.2) are identical and we denoted by $\rho_1, \rho_2, \ldots, \rho_n$ the n solutions which have positive real parts.

Consider the Lundberg's matrices $\mathbf{L}_\delta(\rho_i)$, $i = 1, 2, \ldots, n$. All those matrices are singular, or equivalently all of them have 0 as an eigenvalue. Let $\mathbf{h}_i$ be an eigenvector of $\mathbf{L}_\delta(\rho_i)$ associated to the eigenvalue 0 or, equivalently, let $\mathbf{h}_i$ be a vector in the null space of $\mathbf{L}_\delta(\rho_i)$.

Theorem 4.1. *Let* $\rho_1, \rho_2, \ldots, \rho_m$ *be distinct,* $2 \leq m \leq n$*. Then the eigenvectors* $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_m$ *are linearly independent.*

Proof. By contradiction, suppose that they are linearly dependent. Assume that we can find a subset of l elements of $\{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_m\}$, with $2 \leq l \leq m$, that is linearly dependent and that every subset with $l-1$ elements or less is linearly independent.

Without loss of generality assume that the dependent subset is $\{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{l-1}\}$.

Then there are constants $c_1, c_2, \ldots, c_m$ not all zero such that

$$c_1\mathbf{h}_1 + c_2\mathbf{h}_2 + \cdots + c_l\mathbf{h}_l = \mathbf{0}.$$

Assume that $c_l \neq 0$, then we can write

$$\mathbf{h}_l = \sum_{i=1}^{l-1} \tilde{c}_i \mathbf{h}_i, \quad \tilde{c}_i = -\frac{c_i}{c_l}.$$

Multiplying both sides by $\mathbf{L}_\delta(\rho_l)$ we obtain

$$\begin{aligned} \mathbf{0} &= \mathbf{L}_\delta(\rho_l)\mathbf{h}_l = \mathbf{L}_\delta(\rho_l)\sum_{i=1}^{l-1}\tilde{c}_i\mathbf{h}_i \\ & \sum_{i=1}^{l-1}\tilde{c}_i\mathbf{L}_\delta(\rho_l)\mathbf{h}_i = \sum_{i=1}^{l-1}\tilde{c}_i\tilde{\mathbf{h}}_i, \end{aligned}$$

where $\tilde{\mathbf{h}}_i = \mathbf{L}_\delta(\rho_l)\mathbf{h}_i$, $i = 1, \ldots, l-1$.

Since $\mathbf{h}_i$, $i = 1, \ldots, l-1$ are not eigenvectors of $\mathbf{L}_\delta(\rho_l)$ we have $\tilde{\mathbf{h}}_i \neq \mathbf{0}$, so the vectors $\tilde{\mathbf{h}}_i$ are linearly dependent.

Now the eigenvectors $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{l-1}$ are linearly independent by assumption and they are not in the null space of $\mathbf{L}_\delta(\rho_l)$, therefore $\mathbf{L}_\delta(\rho_l)$ maps them to another set of linearly independent vectors. But this means that $\tilde{\mathbf{h}}_i$ are linearly independent and this is a contradiction.

$\square$

5 The First Time the Surplus Attain a Certain Level

For a barrier level $b \geq u$ define

$$T_b = min\{t \geq 0 : U(t) = b\},$$

to be the first time the surplus reaches level b. For $\delta \geq 0$ define

$$R(u,b) = E[e^{-\delta T_b} | U(0) = u],$$

to be the Laplace transform of T_b. Furthermore define

$$R_{i,j}(u,b) = E_i[e^{-\delta T_b} \mathbb{I}(J(T_b) = j) | U(0) = u],$$

to be the Laplace transform of T_b when the process starts from initial surplus u at state i and reaches the level b at state j. Then

$$R(u,b) = \alpha \mathbf{R}(u,b) \mathbf{e}^\mathsf{T},$$

where $\mathbf{R}(u,b) = (R_{i,j}(u,b))_{i,j=1}^n$.

It follows from Li (2008) that

$$\mathbf{R}(u,b) = e^{-\mathbf{K}(b-u)}, \quad R(u,b) = \alpha e^{-\mathbf{K}(b-u)} \mathbf{e}^\mathsf{T}, \quad u \leq b,$$

where $\mathbf{K}$ is a $n \times n$ matrix that satisfies the following equation

$$c\mathbf{K} = (\delta \mathbf{I} - \mathbf{B}) - \mathbf{b}^\mathsf{T} \alpha \int_0^\infty p(x) e^{-\mathbf{K}x} dx.$$

Assuming that the roots of the fundamental Lundberg's equation with positive real parts $\rho_1, \rho_2, \ldots, \rho_n$ are distinct, Li (2008) shows that

$$\mathbf{K} = \mathbf{H\Delta H^{-1}},$$

where $\mathbf{\Delta} = diag(\rho_1, \rho_2, \ldots, \rho_n)$ and $\mathbf{H} = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n)$. The column vector $\mathbf{h}_i$ is an eigenvector of $L_\delta(\rho_i)$ corresponding to the eigenvalue 0. Then

$$R(u,b) = \alpha \mathbf{H} e^{-\mathbf{\Delta}(b-u)} \mathbf{H^{-1}} \mathbf{e}^\mathsf{T}, \quad u \le b. \tag{5.1}$$

If the roots $\rho_1, \rho_2, \ldots, \rho_n$ are not all distinct then the matrix $\mathbf{H}$ is not invertible and we can not apply formula (5.1) to find $R(u,b)$.

In the case of a double root we propose to replace on the appearances of such root by one of the negative roots of the fundamental Lundberg's equation. There is always one negative root, we denote it by $\rho_0 = -r$ where $r > 0$ is the adjustment coefficient.

Example 5.1. *We continue the last example. Choosing $\beta = 0,67$ the fundamental Lundberg's equation has the following roots*

$$\rho_0 - r = -0.58, \rho_1 = 0.69, \rho_2 = \rho_3 = 2.61,$$

the corresponding eigenvectors are

$$\mathbf{h}_0 = (0.15, 0.49, 0.85), \mathbf{h}_1 = (0.77, 0.47, 0.41), \mathbf{h}_2 = \rho_3 = (0.27, -0.89, 0.36),$$

Therefore

$$\mathbf{H} = \begin{pmatrix} 0.15 & 0.77 & 0.27 \\ 0.49 & 0.47 & -0.89 \\ 0.85 & 0.41 & 0.36 \end{pmatrix}$$

and we apply formula (5.1) *to obtain*

$$R(u,b) = 0.1e^{0.58(b-u)} + 0.93e^{-0.69(b-u)} - 0.034e^{-2.61(b-u)}$$

Remark 5.1. *In the case of double roots we can apply the same method to compute other quantities like the ultimate and finite time ruin probabilities, severity of ruin and its maximum, the expected discounted future dividends, among others.*

6 Conclusions

We studied the fundamental Lundberg's equation to find cases where double roots can arise and for such cases we provided a method to compute the Laplace Transform of the time to reach a certain level. Regarding the Lundberg's Matrix, we gave a proof of the linear independence of the eigenvectors related to different eigenvalues.

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