Further advances on the maximum severity of ruin in an Erlang(n) risk process

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Abstract

For actuarial aplications we consider the Sparre-Andersen risk model when the interclaim times are Erlang(n) distributed. We first address the problem of computing $\chi(u,b)$, the probability that the surplus reaches an upper given level b without first falling below zero, starting from an initial surplus u, $b \geq u \geq 0$. Then, we work on the maximum severity of ruin, if it occurs.

We start by presenting an alternative and improved method to find an expression for calculating $\chi(u,b)$ to that presented by Li (2008). This result will then allow us to find a generalization of Li's (2008) result, considering the natures of the roots of the generalized Lundberg's equation, i.e., whether these roots are distinct or have some multiplicity. For the case when single claim amounts are Erlang distributed we prove that they are always distinct.

Afterwards, we apply our findings above in the computation of the distribution of the maximum severity of ruin, which computation depends on the non-ruin probability, on the roots of the generalized Lundberg's equation, as well as also their nature.

We illustrate and give explicit formulae for Erlang(3) interclaim arrivals with exponentially distributed single claim amounts and Erlang(2) interclaim times with Erlang(2) claim amounts.

Keywords: Sparre-Andersen risk model; Erlang(n) interclaim times; probability of reaching an upper barrier; severity of ruin; maximum severity of ruin.

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1 Introduction

.In the present article we deal with the Sparre–Andersen model

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i,\ t \geq 0,\ u \geq 0,$$

The surplus process starts from an initial capital u, increases in time with a loading factor $c > 0$, and decreases by the quantities X_i , which are random variables and represent the claim amounts that appear randomly in time. We assume that $\{X_i\}$ is a sequence of independent and identically distributed (i.i.d) random variables with common distribution function $P(x)$ and density $p(x)$. Denote by $\mu_k = E[X_1^k]$ the k-th moment of X_i.

The number of claims that occurred before a given time t is represented by $N(t) = \max\{k : W_1 + W_2 + \cdots + W_k \leq t\}$, where the random variables W_i denote interclaim times which are considered i.i.d. and also independent from the X_i .

We assume that the interclaim times are Erlang(n,λ) distributed, therefore with density $k(t) = \dfrac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0, \lambda > 0, n \in \mathbb{N}^+$.

It is assumed a positive loading factor, which means that $cE(W_1) > E(X_1)$. Since $E(W_1) = \dfrac{n}{\lambda}$ this gives the inequality $cn > \lambda\mu_1$.

Now we set some mathematical preliminaries presenting main objects of interest in the Sparre–Andersen model that we will take into consideration.

The time of ruin is denoted by $T = \inf\{t > 0 : U(t) < 0\},\ u \geq 0$, where $T = \infty$ if and only if $U(t) \geq 0 \ \ \forall t > 0$. The ultimate ruin probability is defined as $\Psi(u) = P(T < \infty)$, and the corresponding survival probability is $\Phi(u) = 1 - \Psi(u)$.

Regarding the barrier problem, which is related to the payment of dividends, we denote by $\tau_b = \inf\{t > 0 : U(0) = u, U(t) \geq b\}$ the first time that the surplus upcrosses the level $b \geq u$. The probability that the surplus attains the level b from initial surplus u without first falling below zero is

$$\chi(u, b) = P(T > \tau_b | U(0) = u),$$

with $\xi(u, b) = 1 - \chi(u, b)$ being the probability that ruin occurs from u without the surplus ever reaching b.

If we assume that the surplus process continues after ruin, we denote the time of the first upcross of the surplus through level 0 after ruin occurs by

$T' = \inf\{t : \ t > T, \ U(t) \geq 0\}$. In the interval of time when surplus is at deficit, we define the maximum severity of ruin as

$$M_u = \sup\{|U(t)| : \ U(0) = u, \ T \leq t \leq T'\}, \ u \geq 0.$$

The distribution function of the maximum severity of ruin given that ruin occurs is

$$J(z; u) = P(M_u \leq z \mid T < \infty), \ u, z \geq 0.$$

The probability that ruin occurs and that the deficit at ruin is at most y is $G(u, y) = P(T < \infty, U(T) \geq y)$. For a fixed u, this is a defective distribution function since $\lim_{y \to \infty} G(u, y) = \Psi(u)$, and the corresponding defective density is $g(u, y) = \dfrac{dG(u, y)}{dy}$.

Finally the probability that the maximum deficit occurs at ruin is defined by $P(M_u = |U(T)| \mid T < \infty)$. Picard (1994) showed that

$$P(M_u = |U(T)| \mid T < \infty) = \frac{\int_0^\infty g(u, y)\chi(0, y)dy}{\Psi(u)} \tag{1}$$

In the next section we present some of the mathematical background of the model and write on the motivation. Sections **3** and **4** are the core of this manuscript where we state our main theoretical results. Section **5** is devoted to some particular cases where explicit expressions can easily be found. Finally in the last section we state some concluding remarks.

2 Motivation and mathematical background

The Sparre–Andersen model has been one of the main points of interest in the risk theory in the recent years. Many authors have done a lot of important advances in the topic, either in general or in particular cases.

In this article we want to present some new developments. First of all we present an improved method to find an expression for $\chi(u, b)$.

So far we know from Li & Dickson (2006) that $\chi(u, b)$ satisfies an order n integro-differential equation with n boundary conditions that can be written in the form

$$B(\mathcal{D})v(u) = \int_0^u v(u - y)p(y)dy, \quad u \geq 0, \tag{2}$$

where

$$B(\mathcal{D}) = \left(I - \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n = \sum_{k=0}^{n}(-1)^k\left(\frac{c}{\lambda}\right)^k\binom{n}{k}\mathcal{D}^k$$

and $\mathcal{D}$ is the differential operator.

If we find n linearly independent particular solutions $v_j(u),\ j = 1, \ldots, n$ for this equation, then we have

$$\chi(u, b) = \overrightarrow{v}(u)[V(b)]^{-1}\overrightarrow{e}^T, \tag{3}$$

where $\overrightarrow{v}(u) = (v_1(u), \ldots, v_n(u))$ is a $1\times n$ vector, $(V(b))_{ij} = \left.\frac{d^{i-1}v_j(u)}{du^{i-1}}\right|_{u=b}$ is a $n \times n$ matrix and $\overrightarrow{e} = (1, 0, \ldots, 0)$ is a $1 \times n$ vector.

What we improved is the way to seek for those solutions, depending on the nature of the roots of the generalized Lundberg's equation.

Recall that the generalized Lundberg's equation is $B(s) = \hat{p}(s)$, where $B(s) = (1-(\frac{c}{\lambda})s)^n$. We denote by the numbers $\rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C}$, the only roots of this equation which have positive real parts (there are of course other roots, among which is 0 and $-R$, where $R > 0$ is the adjustment coefficient, see Li & Garrido (2004)).

After that, we apply our results also to find the corresponding expressions for the maximum severity of ruin.

Finally we present some numerical results for two particular cases, the first is Erlang(3) distributed interclaim times with Exponentially distributed claim amounts, and the second is Erlang(2) distributed interclaim times with Erlang(2) distributed claim amounts.

First of all in the subsections **1.1** and **1.2** we set the mathematical background and we explain the reasons that motivates the present work. Sections **2** and **3** are the main core of this article where we state our main theoretical results. Section **4** is devoted to some explicit expressions and finally in section **5** we give the concluding remarks and possible future ways of development.

3 Solutions for the integro-differential equation

In order to obtain the solutions of the integro–differential equation Li (2008) showed that

Theorem 1: If $\rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C}$ are distinct, then we have the following expressions for the $v_j(u)$'s

$$\begin{aligned} v_1(u) &= \Phi(u), \\ v_j(u) &= \sum_{i=1}^{j-1} a_{i,j} \int_0^u \Phi(u-y) e^{\rho_i y} dy, \quad j = 2, 3, \ldots, n, \end{aligned}$$

where $a_{i,j} = -\frac{1}{\prod_{k=1,k\neq i}^{j-1} (\rho_k - \rho_i)}, \quad i = 1, 2, \ldots, j-1.$

We propose new version of Theorem 1 as follows

Theorem 2: If $\rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C}$ are distinct, then we have the following expressions for the $v_j(u)$'s

$$\begin{aligned} v_1(u) &= \Phi(u), \\ v_j(u) &= \int_0^u \Phi(u-y) e^{\rho_{j-1} y} dy, \quad j = 2, 3, \ldots, n. \end{aligned}$$

Proof:

We know from Li (2008) that any solution $v(u)$ of (2) has Laplace transform

$$\hat{v}(s) = \frac{d_v(s)}{B(s) - \hat{p}(s)},$$

where

$$d_v(s) = \sum_{i=0}^{n-1} \left(\sum_{k=i+1}^{n} \binom{n}{k} \left(\frac{-c}{\lambda} \right)^k v^{(k-1-i)}(0) \right) s^i$$

Also as $\Phi(u)$ is solution of (2) with Laplace transform

$$\hat{\Phi}(s) = -\Phi(0) \left(\frac{c}{\lambda} \right)^n \frac{\prod_{i=1}^{n-1} (\rho_i - s)}{B(s) - \hat{p}(s)},$$

so

$$d_\Phi(s) = -\Phi(0) \left(\frac{c}{\lambda} \right)^n \prod_{i=1}^{n-1} (\rho_i - s)$$

Now to see that any function $v_j(u) = \int_0^u \Phi(u-y)e^{\rho_{j-1}y}dy$, with $j = 2, 3, \ldots, n$, is solution of (2) we can easily prove that

$$B(\mathcal{D})v_j(u) = d_\Phi(\rho_{j-1})e^{\rho_{j-1}u} + \int_0^u (B(\mathcal{D})\Phi(u-t))e^{\rho_{j-1}t}dt$$

and that

$$\int_0^u v_j(u-y)p(y)dy = \int_0^u (B(\mathcal{D})\Phi(u-t))e^{\rho_{j-1}t}dt.$$

Since $d_\Phi(\rho_{j-1}) = 0$ we get the desired equality.

So the only remaining thing to prove is that those $v_j(u)$'s are linearly independent.

Suppose that we have a linear combination $\displaystyle\sum_{j=1}^{n} c_j v_j(u) = 0, \quad \forall u \geq 0,$ Consider the case (i) and (ii). We start with (i) and see that

If $c_1 = 0$:

Let $H(t) = \displaystyle\sum_{j=2}^{n} c_j e^{\rho_{j-1}t}$

$$\begin{aligned} \sum_{j=1}^{n} c_j v_j(u) &= \sum_{j=2}^{n} c_j \int_0^u \Phi(u-y)e^{\rho_{j-1}y}dy \\ &= \int_0^u \Phi(u-y)\sum_{j=2}^{n} c_j e^{\rho_{j-1}y}dy \\ &= \Phi * H(u) = 0. \end{aligned}$$

The fact that $\Phi * H(u) = 0, \ \ \forall u \geq 0$ with $\Phi(u) \not\equiv 0$, implies $H(u) \equiv 0$ almost everywhere. But $H(t)$ is a continuously differentiable function, this implies that $c_1 = c_2 = \cdots = c_n = 0$.

For (ii) If $c_1 \neq 0$:

Define $G(t) = \sum_{j=2}^{n} \left(\frac{-c_j}{c_1}\right) e^{\rho_{j-1} t}$, so $\Phi * G(u) = \Phi(u) \;\; \forall u \geq 0$.

Not all the remaining coefficients c_j's can be 0, otherwise $G(t) \equiv 0$. But then $\lim_{u\to+\infty} G(u) = \pm\infty$ depending on the sign of the non zero coefficients. As $\Phi(u)$ is a non–decreasing non–negative function with $\lim_{u\to+\infty} \Phi(u) = 1$, we will have that $\lim_{u\to+\infty} \Phi * G(u) = \pm\infty$, which is a contradiction.
This completes the proof. $\square$

An advantage of theorem 2, is that, since for any complex root ρ of the Lundberg's equation the conjugate $\bar{\rho}$ is also a root, we will have that $v(u) = \int_0^u \Phi(u-y)e^{\rho y}dy$ and its conjugate $\overline{v(u)} = \int_0^u \Phi(u-y)e^{\bar{\rho} y}dy$ are both solutions of (2).

Now we consider the case when there might be multiple roots. It is easy to prove that for the case of Erlang distributed claim amounts the roots are all different, like we can see in the following lemma.

Lemma 1: If the individual claim amounts X_i's have Erlang(m,β) distribution then the generalized Lundberg's equation does not have multiple roots.

Proof: We have $p(x) = \frac{\beta^m x^{m-1} e^{-\beta x}}{(m-1)!}$, since claim amounts are Erlang(m,β) distributed. Define the function

$$f(s) = B(s) - \hat{p}(s) = \left(1 - \left(\frac{c}{\lambda}\right) s\right)^n - \frac{\beta^m}{(s+\beta)^m}.$$

Then the roots of the generalized Lundberg's equation are all the solutions of $f(s) = 0$.

These solutions are

$$\rho_1, \rho_2, \ldots, \rho_{n-1}, 0, -R_1, -R_2, \ldots, -R_m,$$

where $Re(\rho_i) > 0$, $Re(R_j) > 0$, and $0 < R_1 < \beta$ is the adjustment coefficient.

Let $g(s) = (1 - (\frac{c}{\lambda})s)^n (s+\beta)^m - \beta^m$. Then $f(s) = 0$ and $g(s) = 0$ have the same set of solutions. The derivative of g is

$$g'(s) = \left(1 - \left(\frac{c}{\lambda}\right) s\right)^{n-1} (s+\beta)^{m-1} \left(-(m+n)\left(\frac{c}{\lambda}\right) s + m - n\left(\frac{c}{\lambda}\right)\beta\right).$$

The roots of $g'(s)$ are

$$\frac{\lambda}{c}, \ -\beta, \quad \text{and} \quad s_0 = \frac{m - n(\frac{c}{\lambda})\beta}{(m+n)(\frac{c}{\lambda})}.$$

Since $-\beta < s_0 < 0$ and $s_0 \neq -R_1$, none of those roots are roots of $g(s)$.

This implies that $g(s)$ and therefore $f(s)$ don't have multiple roots. $\square$

Despite of the last lemma, it hasn't been proven yet that for every possible phase–type distribution that we could choose as claim distribution, the resulting roots $\rho_1, \rho_2, \ldots, \rho_{n-1}$ would be all different. So we state some results for possible cases when we could find multiple roots.

First of all, suppose that we have one root with multiplicity $n-1$.

Theorem 3: If $\rho_1 = \rho_2 = \ldots = \rho_{n-1} = \rho$, then we have the following expressions for the $v_j(u)$'s

$$\begin{aligned} v_1(u) &= \Phi(u), \\ v_j(u) &= \int_0^u \Phi(u-y) y^{j-2} e^{\rho y} dy, \quad j = 2, 3, \ldots, n. \end{aligned}$$

Proof: It can be easily proven that

$$\begin{aligned} B(\mathcal{D}) v_j(u) &= \sum_{l=0}^{j-2} \binom{j-2}{l} d_{\Phi}^{(l)}(\rho) u^{j-2-l} e^{\rho u} + \\ & \sum_{k=0}^{n} \binom{n}{k} \left(\frac{-c}{\lambda}\right)^k \int_0^u \Phi^{(k)}(u-t) t^{j-2} e^{\rho t} dt, \end{aligned}$$

and that

$$\int_0^u v_j(u-y) p(y) dy = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{-c}{\lambda}\right)^k \int_0^u \Phi^{(k)}(u-t) t^{j-2} e^{\rho t} dt.$$

Since $d_\Phi(s) = -\Phi(0)\left(\frac{c}{\lambda}\right)^n (\rho - s)^{n-1}$, then all the derivatives $d_\Phi^{(l)}(\rho)$, $l = 0, \dots, n-2$, are zero, and we get the desired equality as before.

To see the linear independence of the $v_j(u)$'s we can use a slight modification of the proof given for the Theorem 2. $\square$

Consider now a case when there is one root with multiplicity $k-1$, this is, $\rho_1 = \rho_2 = \dots = \rho_{k-1} = \rho$, while the remaining $\rho_k, \rho_{k+1}, \dots, \rho_{n-1}$ have multiplicity 1. We have

Theorem 4: Under the conditions described above, we have the following expressions for the $v_j(u)$'s

$$\begin{aligned} v_1(u) &= \Phi(u), \\ v_j(u) &= \int_0^u \Phi(u-y) y^{j-2} e^{\rho y} dy, \quad j = 2, 3, \dots, k, \\ v_j(u) &= \int_0^u \Phi(u-y) e^{\rho_{j-1} y} dy, \quad j = k+1, \dots, n. \end{aligned}$$

Finally, we can show the general case. Suppose that we have k different roots, $\rho_1, \rho_2, \dots, \rho_k$, where the root ρ_i has multiplicity m_i and

$$\sum_{i=1}^{k} m_i = n - 1 \quad \text{or} \quad \sum_{i=1}^{k} m_i + 1 = n.$$

Theorem 5: Under the conditions described above, we have the following expressions for the $v_j(u)$'s

$$\begin{aligned}
v_1(u) &= \Phi(u), \\
v_j(u) &= \int_0^u \Phi(u-y)y^{j-2}e^{\rho_1 y}dy, \quad j=2,3,\ldots,m_1+1, \\
v_j(u) &= \int_0^u \Phi(u-y)y^{j-m_1-2}e^{\rho_2 y}dy, \quad j=m_1+2,\ldots,m_1+m_2+1, \\
v_j(u) &= \int_0^u \Phi(u-y)y^{j-(m_1+m_2)-2}e^{\rho_3 y}dy, \\
& \quad j=m_1+m_2+2,\ldots,m_1+m_2+m_3+1, \\
&\vdots \\
v_j(u) &= \int_0^u \Phi(u-y)y^{j-(m_1+\cdots+m_{k-1})-2}e^{\rho_k y}dy, \\
& \qquad j=\sum_{i=1}^{k-1} m_i+2,\ldots,\sum_{i=1}^{k} m_i+1=n.
\end{aligned}$$

With some small modifications to the proofs of Theorems 2 and 3 we can obtain the proofs of the Theorems 4 and 5.

4 The maximum severity of ruin

In the previous section we have shown how to obtain the solutions of the integro–differential equation in different situations, depending on the nature of the roots of the generalized Lundberg's equation.

Now we will use these results to obtain the maximum severity of ruin.

We will find an expression for the maximum severity of ruin which only depends on the non ruin probability $\Phi(u)$ and the claim amounts distribution.

From Dickson (2005) and (3) we know that the distribution of the maximum severity of ruin $J(z;u)$ can be expressed as:

$$J(z;u) = \frac{1}{1-\Phi(u)} \int_0^z g(u,y)(v_1(z-y),\ldots,v_n(z-y))dy[V(z)]^{-1}\overrightarrow{e}^T \quad (4)$$

If we denote by

$$\begin{aligned}
\overrightarrow{h}(z,u) &= \int_0^z g(u,y)(v_1(z-y),\ldots,v_n(z-y))dy \\
&= \left(\int_0^z g(u,y)v_1(z-y)dy,\ldots,\int_0^z g(u,y)v_n(z-y)dy\right) \\
&= (h_1(z,u),\ldots,h_n(z,u))
\end{aligned}$$

then we only have to find an expression for every component of $\overrightarrow{h}(z,u)$.

– *For the case in theorem 2*

In a similar way as it is done in Li (2008) we get

($*$) For $j = 1$:

$$\int_0^z g(u,y)v_1(z-y)dy = \Phi(u+z) - \Phi(u) \tag{5}$$

($*$) And for $j = 2,\ldots,n$:

$$\begin{aligned}
\int_0^z g(u,y)v_j(z-y)dy &= \int_0^z g(u,y)\int_0^{z-y}\Phi(z-y-x)e^{\rho_{j-1}x}dxdy \\
&\quad \int_0^z e^{\rho_{j-1}x}[\Phi(u+(z-x))-\Phi(u)]dx
\end{aligned}$$

– *For the general case in theorem 5*

($*$) For $j = 1$:

$$\int_0^z g(u,y)v_1(z-y)dy = \Phi(u+z) - \Phi(u)$$

(∗) For $j = 2, \ldots, m_1 + 1$:

$$\begin{aligned}
\int_0^z g(u,y)v_j(z-y)dy &= \int_0^z g(u,y)\int_0^{z-y}\Phi(z-y-x)x^{j-2}e^{\rho_1 x}dxdy \\
& \quad (\overline{y} = z-y,\ y = z-\overline{y},\ d\overline{y} = -dy) \\
&= \int_0^z g(u,z-\overline{y})\int_0^{\overline{y}}\Phi(\overline{y}-x)x^{j-2}e^{\rho_1 x}dxd\overline{y} \\
& \quad (0 \le x \le z\ \rightarrow\ x \le y \le z) \\
&= \int_0^z x^{j-2}e^{\rho_1 x}\int_x^z g(u,z-\overline{y})\Phi(\overline{y}-x)d\overline{y}dx \\
& \quad (t = z-\overline{y},\ \overline{y} = z-t,\ dt = -d\overline{y}) \\
&= \int_0^z x^{j-2}e^{\rho_1 x}\left[\int_0^{z-x} g(u,t)\Phi((z-x)-t)dt\right]dx
\end{aligned}$$

By (5) we have $\int_0^{z-x} g(u,t)\Phi((z-x)-t)dt = \Phi(u+(z-x)) - \Phi(u)$, so

$$\int_0^z g(u,y)v_j(z-y)dy = \int_0^z x^{j-2}e^{\rho_1 x}[\Phi(u+(z-x)) - \Phi(u)]dx$$

(∗) For $j = m_1 + 2, \ldots, m_1 + m_2 + 1$:
In a similar way we get

$$\int_0^z g(u,y)v_j(z-y)dy = \int_0^z x^{j-m_1-2}e^{\rho_2 x}[\Phi(u+(z-x)) - \Phi(u)]dx$$

(∗) Finally for $j = \sum_{i=1}^{k-1} m_i + 2, \ldots, \sum_{i=1}^{k} m_i + 1 = n$:

$$\int_0^z g(u,y)v_j(z-y)dy = \int_0^z \left[x^{j-(m_1+\cdots+m_{k-1})-2}e^{\rho_k x}\right][\Phi(u+(z-x)) - \Phi(u)]dx$$

In this way we get the desired expression for (4).

5 Explicit expressions

In the section the aim is to determine the moments of the maximum severity of ruin as well as the probability that the maximum severity occurs at ruin. Li (2008) considered those moments for Erlang(2) interclaim times and exponential claims.
Here we will present the formulas and some numerical calculations obtained for two other cases:
– Interclaim arrivals Erlang(3,λ) distributed, and claim amounts Exponential(β) distributed. For simplification we will denote this case by Erlang(3) – Exponential.
– Interclaim arrivals Erlang(2,λ) distributed, and claim amounts Erlang(2,β) distributed. We will denote this case by Erlang(2) – Erlang(2).

5.1 Erlang(3) – Exponential case

Considering the safety loading $c = \dfrac{(1+\theta)\lambda}{3\beta}$ with $\theta > 0$, the generalized Lundberg's equation takes the form

$$\left(1 - \left(\frac{c}{\lambda}\right) s\right)^3 - \frac{\beta}{(s+\beta)} = 0,$$

which has four roots: $0, \rho_1, \rho_2$ and $-R$, where $0 < R < \beta$ is the adjustment coefficient, ρ_1, ρ_2 are complex roots with positive real parts and $\rho_2 = \overline{\rho_1}$.

The 3 solutions for the integro–differential equation (2) are

$$\begin{aligned}
\Phi(u) &= 1 - \left(1 - \frac{R}{\beta}\right) e^{-Ru} \\
v_2(u) &= \frac{-1}{\rho_1} + \frac{\beta - R}{\beta(R+\rho_1)} e^{-Ru} + \frac{R(\beta+\rho_1)}{\rho_1\beta(R+\rho_1)} e^{\rho_1 u} \\
v_3(u) &= \frac{-1}{\rho_2} + \frac{\beta - R}{\beta(R+\rho_2)} e^{-Ru} + \frac{R(\beta+\rho_2)}{\rho_2\beta(R+\rho_2)} e^{\rho_2 u}
\end{aligned}$$

5.1.1 The moments of M_u

After calculating (4) we get

$$1 - J(z;u) = \frac{\alpha e^{-Rz}}{1 - \gamma e^{-(\rho_1+R)z} - \delta e^{-(\rho_2+R)z} - \eta e^{-Rz}},$$

where

$$\alpha = \frac{R(R+\rho_1)(R+\rho_2)}{\beta(\beta+\rho_1)(\beta+\rho_2)} \qquad \gamma = -\frac{R(\beta-R)(R+\rho_2)}{\rho_1(\beta+\rho_1)(\rho_2-\rho_1)}$$
$$\delta = \frac{R(\beta-R)(R+\rho_1)}{\rho_2(\beta+\rho_2)(\rho_2-\rho_1)} \qquad \eta = \frac{(\beta-R)(R+\rho_1)(R+\rho_2)}{\beta\rho_1\rho_2}$$

with $0 < \alpha < 1$, $\delta = \overline{\gamma}$ and $0 < \eta = 1 - \alpha - \gamma - \delta$.

Note that this expression for $J(z;u)$ is independent from u.

Now we compute the moments of M_u given that ruin occurs by the formula

$$\begin{aligned} E(M_u^r|T<\infty) &= r\int_0^\infty z^{r-1}(1-J(z;u))dz \\ &= r\alpha\int_0^\infty \frac{z^{r-1}e^{-Rz}}{1-\gamma e^{-(\rho_1+R)z}-\delta e^{-(\rho_2+R)z}-\eta e^{-Rz}}\,dz, \quad (6) \end{aligned}$$

for $r \geq 1$.
Since $|\gamma e^{-(\rho_1+R)z} + \delta e^{-(\rho_2+R)z} + \eta e^{-Rz}| < 1$ we can write

$$1 - J(z;u) = \alpha e^{-Rz}\sum_{k=0}^{\infty}(\gamma e^{-(\rho_1+R)z} + \delta e^{-(\rho_2+R)z} + \eta e^{-Rz})^k$$

and then

$$E(M_u^r|T<\infty) = \alpha r! \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{k-j}\binom{k}{j}\binom{k-j}{l}\frac{\eta^j\gamma^l\delta^{k-j-l}}{(R(k+1)+\rho_1 l+\rho_2(k-j-l))^r},$$

However, after computing explicit values in software like `Mathematica` the last formula takes much more time to produce results than formula (6).

Now choosing $\beta = 1$, $\lambda = 3$ and $c = 1 + \theta$ we evaluate formula (6) for some values of θ with $r = 1$ and compare with Li (2008) results.

Table 1: The values of $E(M_u)$ and $s.d.(M_u)$ for $n = 1, 2, 3$; $m = 1$.

	n = 1		n = 2		n = 3	
	m = 1		m = 1		m = 1	
θ	$E(M_u)$	$s.d.(M_u)$	$E(M_u)$	$s.d.(M_u)$	$E(M_u)$	$s.d.(M_u)$
0.05	3.197	7.324	2.474	5.532	2.236	4.933
0.1	2.638	5.007	2.063	3.805	1.875	3.404
0.15	2.342	4.015	1.848	3.069	1.687	2.754
0.2	2.150	3.443	1.709	2.646	1.567	2.381
0.25	2.012	3.064	1.611	2.368	1.481	2.136
0.3	1.906	2.792	1.536	2.169	1.416	1.962

>From the table we observe that the mean and the standard deviation of M_u decrease as θ increases for the 3 cases. This was expected since an increase in θ means an increase in our positive safety loading c, which will give faster grow of the surplus per unit of time. Also we note that for a fixed θ the mean and the standard deviation of M_u decrease as n increases. The reason for this is that for higher values of n with fixed m we are increasing the expected value of the interclaim times, which is given by $E(W_i = \frac{n}{\lambda})$, so in average we will get claims in longer intervals of time.

5.1.2 The probability that the maximum severity occurs at ruin

Due to the memory–less property of the exponential distribution we have that $g(u, y) = \Psi(u)p(y)$. Hence from (1)

$$\begin{aligned} P(M_u = |U(T)| \mid T < \infty) &= \frac{\int_0^\infty g(u,y)\chi(0,y)dy}{\Psi(u)} \\ &\int_0^\infty \chi(0,y)p(y)dy. \end{aligned}$$

Now from (3) we get, for $u = 0$

$$\chi(0,y) = \left(\frac{R}{\beta}\right) \frac{1 + \frac{\rho_1\gamma}{R} e^{-(\rho_1+R)y} + \frac{\rho_2\gamma}{R} e^{-(\rho_2+R)y}}{1 - \gamma e^{-(\rho_1+R)y} - \delta e^{-(\rho_2+R)y} - \eta e^{-Ry}}.$$

So

$$\begin{aligned} P(M_u = |U(T)| \mid T < \infty) &= \left(\frac{R}{\beta}\right) \int_0^\infty \frac{1 + \frac{\rho_1\gamma}{R} e^{-(\rho_1+R)y} + \frac{\rho_2\gamma}{R} e^{-(\rho_2+R)y}}{1 - \gamma e^{-(\rho_1+R)y} - \delta e^{-(\rho_2+R)y} - \eta e^{-Ry}} \beta e^{-\beta y} dy \\ &= \int_0^\infty \frac{R + \rho_1\gamma e^{-(\rho_1+R)y} + \rho_2\gamma e^{-(\rho_2+R)y}}{1 - \gamma e^{-(\rho_1+R)y} - \delta e^{-(\rho_2+R)y} - \eta e^{-Ry}} e^{-\beta y} dy \qquad (7) \end{aligned}$$

We can write this probability in the form of a series as we did before for the moments of M_u to obtain the following

$$\begin{aligned} P(M_u = |U(T)| \mid T < \infty) &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{l=0}^{k-j} \binom{k}{j} \binom{k-j}{l} \eta^j \gamma^l \delta^{k-j-l} \cdot \\ \left[\frac{R}{\beta + Rk + \rho_1 l + \rho_2(k-j-l)}\right. &+ \frac{\rho_1\gamma}{\beta + R(k+1) + \rho_1(l+1) + \rho_2(k-j-l)} + \\ &+ \left.\frac{\rho_2\gamma}{\beta + R(k+1) + \rho_1 l + \rho_2(k-j-l+1)}\right], \end{aligned}$$

In the same way, after computing explicit values with `Mathematica` the last formula takes much more time to produce results than formula (7).

Choosing the same values of λ, β and θ as before we evaluate (7) to get the following table

Table 2: Probability that the maximum deficit occurs at ruin for $n = 3$, $m = 1$.

	n = 3
θ	$P(M_u = \|U(T)\| \mid T < \infty)$
0.05	0.735
0.1	0.752
0.15	0.768
0.2	0.782
0.25	0.795
0.3	0.808

From the table we conclude that the probability that the maximum deficit occurs at ruin increases as θ increases. This means that for bigger safety loadings c is less likely that the surplus will drop to lower levels of deficit after the time of ruin.

5.2 Erlang(2) – Erlang(2) case

Considering the safety loading $c = \dfrac{(1+\theta)\lambda}{\beta}$ with $\theta > 0$, the generalized Lundberg's equation takes the form

$$\left(1 - \left(\frac{c}{\lambda}\right) s\right)^2 - \frac{\beta^2}{(s+\beta)^2} = 0,$$

which has four real roots: $0, -R_1, -R_2$ and ρ, where $0 < R_1 < \beta$ is the adjustment coefficient and $R_2, \rho > \beta$.

The 2 solutions for the integro–differential equation (2) are

$$\begin{aligned}
\Phi(u) &= 1 - \frac{R_2(\beta - R_1)^2}{\beta^2(R_2 - R_1)} e^{-R_1 u} - \frac{R_1(\beta - R_2)^2}{\beta^2(R_1 - R_2)} e^{-R_2 u} \\
v_2(u) &= \frac{-1}{\rho} + \frac{R_1 R_2(\beta + \rho)^2}{\beta^2 \rho(\rho + R_1)(\rho + R_2)} e^{\rho u} + \frac{R_2(\beta - R_1)^2}{\beta^2(R_2 - R_1)(\rho + R_1)} e^{-R_1 u} + \\
&\quad \frac{R_1(\beta - R_2)^2}{\beta^2(R_1 - R_2)(\rho + R_2)} e^{-R_2 u}
\end{aligned}$$

5.2.1 The moments of M_u

In this case the formula that we get from (4) is not independent from u, we write it in the following way

$$J(z;u) = \frac{1}{\Psi(u)}\left[\frac{R_2(\beta-R_1)^2}{\beta^2(R_2-R_1)}e^{-R_1u}J_1(z;u)+\frac{R_1(\beta-R_2)^2}{\beta^2(R_1-R_2)}e^{-R_2u}J_2(z;u)\right],$$

and so

$$1-J(z;u)=\frac{1}{\Psi(u)}\left[\frac{R_2(\beta-R_1)^2}{\beta^2(R_2-R_1)}e^{-R_1u}(1-J_1(z;u))+\frac{R_1(\beta-R_2)^2}{\beta^2(R_1-R_2)}e^{-R_2u}(1-J_2(z;u))\right]$$

where

$$J_1(z;u) = \frac{1-\gamma_1e^{-(\rho+R_1)z}-\gamma_2e^{-(\rho+R_2)z}-(1-\gamma_1)e^{-R_1z}-\tau_1e^{-R_2z}-\omega_1e^{-(\rho+R_1+R_2)z}}{1-\gamma_1e^{-(\rho+R_1)z}-\gamma_2e^{-(\rho+R_2)z}-\delta_1e^{-R_1z}-\delta_2e^{-R_2z}-\eta e^{-(\rho+R_1+R_2)z}}$$

$$J_2(z;u) = \frac{1-\gamma_1e^{-(\rho+R_1)z}-\gamma_2e^{-(\rho+R_2)z}-\tau_2e^{-R_1z}-(1-\gamma_2)e^{-R_2z}-\omega_2e^{-(\rho+R_1+R_2)z}}{1-\gamma_1e^{-(\rho+R_1)z}-\gamma_2e^{-(\rho+R_2)z}-\delta_1e^{-R_1z}-\delta_2e^{-R_2z}-\eta e^{-(\rho+R_1+R_2)z}}$$

and

$$\gamma_1=-\frac{R_1(\beta-R_1)^2(\rho+R_2)}{\rho(R_2-R_1)(\beta+\rho)^2} \qquad \gamma_2=-\frac{R_2(\beta-R_2)^2(\rho+R_1)}{\rho(R_1-R_2)(\beta+\rho)^2}$$

$$\delta_1=\frac{R_2(\beta-R_1)^2(\rho+R_1)}{\beta^2\rho(R_2-R_1)} \qquad \delta_2=\frac{R_1(\beta-R_2)^2(\rho+R_2)}{\beta^2\rho(R_1-R_2)}$$

$$\tau_1=\frac{R_1(\beta-R_2)^2(\rho+R_2)}{\rho(R_1-R_2)(\beta+\rho)^2} \qquad \tau_2=\frac{R_2(\beta-R_1)^2(\rho+R_1)}{\rho(R_2-R_1)(\beta+\rho)^2}$$

$$\omega_1=-\frac{(\beta-R_2)^2}{(\beta+\rho)^2} \qquad \omega_2=-\frac{(\beta-R_1)^2}{(\beta+\rho)^2}$$

$$\eta=-\frac{(\beta-R_1)^2(\beta-R_2)^2}{\beta^2(\beta+\rho)^2} \qquad \alpha=\frac{R_1R_2(\rho+R_1)(\rho+R_2)}{\beta^2(\beta+\rho)^2},$$

with $0<\alpha<1$ and $\eta=1-\alpha-\gamma_1-\gamma_2-\delta_1-\delta_2$.

In the same way we compute the moments of M_u given that ruin occurs

$$
\begin{aligned}
E(M_u^r|T<\infty) &= r\int_0^\infty z^{r-1}(1-J(z;u))dz \\
&= \frac{r}{\Psi(u)}\left[\frac{R_2(\beta-R_1)^2}{\beta^2(R_2-R_1)}e^{-R_1 u}\int_0^\infty z^{r-1}(1-J_1(z;u))dz+\right. \\
&\quad \left.\frac{R_1(\beta-R_2)^2}{\beta^2(R_1-R_2)}e^{-R_2 u}\int_0^\infty z^{r-1}(1-J_2(z;u))dz\right], \qquad (8)
\end{aligned}
$$

for $r \geq 1$.

Also since

$$
|\gamma_1 e^{-(\rho+R_1)z} + \gamma_2 e^{-(\rho+R_2)z} + \delta_1 e^{-R_1 z} + \delta_2 e^{-R_2 z} + \eta e^{-(\rho+R_1+R_2)z}| < 1,
$$

Like we did before, we found an expression for the moments of the maximum severity in the form of a series. The formula is very long and we will omit it. For computing purposes in `Mathematica` are using formula (8).

Now choosing $\beta = 1$, $\lambda = 1$ and $c = 1+\theta$ we evaluate formula (8) for some values of θ with $r = 1$ and compare with Li (2008) results.

Table 3: The values of $E(M_u)$ and $s.d.(M_u)$ for $n = 1, 2; : m = 1$ and $n = m = 2$.

θ	n = 2 m = 1 $E(M_u)$	$s.d.(M_u)$	n = 2 m = 2 $E(M_u)$	$s.d.(M_u)$
0.05	2.474	5.532	3.279	7.137
0.1	2.063	3.805	2.759	4.911
0.15	1.848	3.069	2.485	3.959
0.2	1.709	2.646	2.307	3.411
0.25	1.611	2.368	2.179	3.049
0.3	1.536	2.169	2.082	2.791

From the table we observe that the mean and the standard deviation of M_u decrease as θ increases for the 3 cases. This was expected since an increase in θ means an increase in our positive safety loading c, which will give faster grow of the surplus per unit of time. Also we note that for a fixed θ the mean and the standard deviation of M_u are higher in the Erlang(2)

– Erlang(2) case than in the Erlang(2) – Exponential case. The reason for this is that for higher values of m with fixed n we are increasing the expected value of the claim amounts, which is given by $E(X_i = \frac{m}{\beta})$, so we are increasing the average size of the claims that we will receive.

5.2.2 The probability that the maximum severity occurs at ruin

In this case the Erlang(2) distribution is not memory–less, so we don't have $g(u, y) = \Psi(u)p(y)$ as in the Erlang(3)–Exponential case.

After computing $g(u, y)$ we get

$$g(u,y) = \frac{(\beta - R_1)^2}{R_2 - R_1}(1-(\beta-R_2)y)e^{-\beta y}e^{-R_1 u} + \frac{(\beta - R_2)^2}{R_1 - R_2}(1-(\beta-R_1)y)e^{-\beta y}e^{-R_2 u}$$

From (3) we get, for $u = 0$

$$\chi(0, y) = \left(\frac{R_1 R_2}{\beta^2}\right) \frac{1 + \frac{\rho\gamma_1}{R_1}e^{-(\rho+R_1)y} + \frac{\rho\gamma_2}{R_2}e^{-(\rho+R_2)y}}{1 - \gamma_1 e^{-(\rho+R_1)z} - \gamma_2 e^{-(\rho+R_2)z} - \delta_1 e^{-R_1 z} - \delta_2 e^{-R_2 z} - \eta e^{-(\rho+R_1+R_2)z}}.$$

So

$$P(M_u = |U(T)| \mid T < \infty) \quad = \quad \frac{\int_0^\infty g(u,y)\chi(0,y)dy}{\Psi(u)} \tag{9}$$

The formula that we obtain in (9) is too large and we omit it.

Choosing the same values of λ, β and θ as before we evaluate (9) to get the following table

Table 4: Probability that the maximum deficit occurs at ruin for $n = 2$, $m = 2$.

θ	n = m = 2 $P(M_u = \|U(T)\| \mid T < \infty)$
0.05	0.730
0.1	0.745
0.15	0.759
0.2	0.772
0.25	0.784
0.3	0.795

From the table we conclude that the probability that the maximum deficit occurs at ruin increase as θ increases. Like in Section **5.1.2**, this means that for bigger safety loadings c is less likely that the surplus will drop to lower levels of deficit after the time of ruin.

6 Summary and Conclusions

In this article we have shown, based on the techniques provided by Li (2008), a new method to find expressions for the distribution of the maximum severity of ruin in the Sparre–Andersen model with Erlang(n) interclaim times. Those expressions depend exclusively on the non–ruin probability and the claim amounts distribution.

Following the approach of this article other explicit formulas for the moments of the maximum severity of ruin can be found for particular cases of interest, like Erlang(n) - Erlang(m) for higher values of m and n, since for those cases the expression for the non–ruin probability is available.

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