Optimal trading under coherent comonotonic risk measures*

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Abstract: This paper deals with the optimal risk trading from the point of view of an individual who rates his position using a coherent comonotonic risk measure, assuming that the market price is also coherent and comonotonic.

We obtain a simple and intuitive explicit solution in terms of Kusuoka representations.

KEY WORDS: Coherent risk measures, Risk-adjusted risk measures, Optimal trading, Optimal risk cedence.

1 Introduction

The optimal sharing of risk between economic agents is a prominent topic in economics and management sciences at least since the works of Arrow (1963), Borch (1962), Bühlmann (1984) and Bühlmann and Jewell (1979).

A large corpus of publications dealing with this type of problems is set in a game-theoretical framework – see, among others, Burgert and Rüschendorf (2008), Burgert and Rüschendorf (2006), Filipović and Svindland (2008), Heath and Ku (2004), Jouini et al. (2008), Kaina and Rüschendorf (2009), Kiesel and Rüschendorf (2010), Ludkovski and Rüschendorf (2008), and references therein. Most of these works use duality methods to characterize Pareto equilibria in a market containing a number of agents possessing risky assets and seeking to trade them with the other agents in order to improve individual risk measures. These risk measures are usually assumed to satisfy some desirable properties (e.g., coherence in the sense of Artzner et al. (1999)), but are otherwise generic. Not surprisingly due to this level of generality, the characterizations obtained

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by these authors are general necessary and sufficient conditions of optimality that are not very informative about explicit forms for the optimal trade.

There is also a large number of publications dealing with the topic in an optimization framework – see Balbás et al. (2009), Bernard and Tian (2009), Cai and Tan (2007), Cai et al. (2008), Centeno and Guerra (2010), Gajek and Zagrodny (2004a), Gajek and Zagrodny (2004b), Guerra and Centeno (2010), Guerra and Centeno (2008), Kaluszka (2005a), Kaluszka (2005b), Kaluszka and Okolewski (2008), Promislow and Young (2005) and Rockafellar and Uryasev (2000), and references therein. Most of these publications are specifically concerned with insurance/reinsurance problems. They assume that there is some mechanism for pricing risk (i.e., a premium calculation principle), the agent owns some risk and seeks to buy some insurance, in order to minimize some risk measure, net of insurance premium. In most of these works, both the premium calculation principle and the agent's risk measure are specified within some more or less narrow set and hence it is possible to obtain structural characterization of the optimal insurance/reinsurance (e.g., stop-loss, stop-loss with ceiling, etc.).

In the present paper we assume an external pricing mechanism (a functional) and consider an agent who owns a risk an seeks to trade in order to maximize a risk measure. Thus, we are considering an optimization problem. However, both the pricing functional and the agent's risk measure are assumed to be generic coherent comonotonic cash-invariant functionals. Also, the type of transactions we consider is very general. We show how to narrow them down to insurance contracts, but a priori they include many other types of contracts, including financial derivatives like options. Kusuoka (2001) gave a representation of coherent comonotonic cash-invariant functionals. It turns out that this representation allows for direct optimization methods. Thus, we obtain an explicit solution for the optimization problem. Further, this solution is quite intuitive from the economic point of view.

This paper is organized as follows. In Section 2, we give a full definition of the problem, presenting and discussing the underlying assumptions. We give the solution in the form of Theorem 2. In Section 3, we apply the Theorem on two simple examples (a problem of reinsurance and a problem of asset returns). Section 4 contains the proof of Theorem 2.

2 Problem setting and solution

By a risk, we understand a real random variable $X : \Omega \mapsto \mathbb{R}$, defined in some probability space $(\Omega, \mathcal{F}, P)$. X represents some monetary quantity that will materialize at a given time in the future, but is not possible to forecast exactly from the present. For example, X may represent the value of a portfolio at some given moment in the future, the sum of revenues and liabilities generated in some future period by a contract or a set of contracts, the amount of claims placed under an insurance policy during a certain period, etc..

Let $\mathcal{X}$ denote the set of all random variables, i.e., the space of all $\mathcal{F}$-measurable real functions with domain Ω. A risk measure is a real functional with domain in some nonempty subset of $\mathcal{X}$. Following the approach of Artzner et al. (1999), we assume that an agent who rates risks by a measure ρ_A will accept to take a given risk X from other agent only if he receives a compensation (premium) greater or equal than $\rho_A(X)$, if this value is non-negative. If $\rho_A(X)$ is negative, then the agent is willing to purchase the risk X for any price not exceeding $-\rho_A(X)$. Thus, $-\rho_A(X)$ is the agent's biding price for risk X, while $\rho_A(-X)$ is the agent's ask price. Conversely, consider a set of risks that are "tradable" in the sense that each of them can be procured in a market for some (positive or negative) price. The map assigning to each risk the corresponding price can be equated to a risk measure $X \mapsto \rho_M(-X)$.

In this paper we consider an agent who owns a risk Y and wants to trade in order to minimize a personal risk measure ρ_A. We call Y the "**standing risk**". The agent has access to a market where any risk Z lying in a certain class $\mathcal{Z} \subset \mathcal{X}$ can be traded for a price given by the market risk measure $\rho_M(-Z)$.

We are not concerned with the mechanism by which this price is formed and assume that the agent knows the "princing rule", i.e., he knows the functional ρ_M, but has no influence over it. This means that our agent is small compared to the size of the market or the size of his possible trading partner(s). For example, he may be a small trader operating in a competitive market. In this case ρ_M is not expected to coincide with any particular agent's risk measure. Instead, it will be a risk measure implicit in the collective behavior of all the agents in the market. Another example is an individual trader facing a much larger monopolistic partner. In this case ρ_M can be taken to represent the monopolist's own risk measure.

Artzner et al. (1999) introduced the concept of coherent risk measures:

Definition 1 *A risk measure ρ defined in some suitable convex set of random variables $\mathcal{Z}$ is said to be coherent if it satisfies the following conditions:*

Translation invariance: *for any risk $X \in \mathcal{Z}$ and any constant $c \in \mathbb{R}$: $\rho(X - c) = \rho(X) + c$;*

Subadditivity: *for any risks $X_1,\ X_2 \in \mathcal{Z}$: $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$;*

Positive homogeneity: *for any risk $X \in \mathcal{Z}$ and any constant $\lambda \geq 0$: $\rho(\lambda X) = \lambda \rho(X)$;*

Monotonicity: *for any risks $X_1,\ X_2 \in \mathcal{Z}$ such that $\Pr\{X_1 \geq X_2\} = 1$: $\rho(X_1) \leq \rho(X_2)$.* $\square$

We also need to introduce the notion of comonotonic risk measure:

Definition 2 *Two random variables $X_1, X_2 : \Omega \mapsto \mathbb{R}$ are said to be comonotone if*

$$(P \times P)\{(\omega_1, \omega_2) \in \Omega^2 : (X_1(\omega_1) - X_1(\omega_2))(X_2(\omega_1) - X_2(\omega_2)) \geq 0\} = 1.$$

A risk measure $\rho : \mathcal{Z} \mapsto \mathbb{R}$ is said to be comonotonic if

$$\rho(X_1 + X_2) = \rho(X_1) + \rho(X_2)$$

for every comonotone pair $X_1,\ X_2 \in \mathcal{Z}$*.* $\square$

Kusuoka (2001) proved the following result (here presented in a slightly different but equivalent formulation):

Theorem 1 *Suppose* $(\Omega, \mathcal{F}, P)$ *is atom-less and consider a functional* $\rho : L_\infty(\Omega, \mathcal{F}, P) \mapsto \mathbb{R}$*. The following assertions are equivalent:*

(i) ρ *is a law invariant comonotonic coherent risk measure with the Fatou property;*

(ii) *There is a continuous nondecreasing concave function* $w : [0,1] \mapsto [0,1]$*, with* $w(0) = 0$*, such that*

$$\begin{aligned}\rho(X) =& (1 - w(1)) \left(\mathrm{esssup}(X) - \mathrm{essinf}(X)\right) + \\ &+ \int_{-\infty}^{0} w(\Pr\{X < t\})\, dt + \int_{0}^{\mathrm{esssup}(X)} \left(w(\Pr\{X < t\}) - 1\right)\, dt,\end{aligned} \tag{1}$$

For every $X \in L_\infty(\Omega, \mathcal{F}, P)$*.* $\square$

Jouini et al. (2006) proved that, provided $(\Omega, \mathcal{F}, P)$ is atom-less, all law invariant coherent risk measures have the Fatou property, and hence this condition can be omitted from Theorem 1.

Our result is formulated in terms of Kusuoka representations (1) for the risk measures ρ_A, ρ_M. Thus, the following is a natural technical assumption:

Assumption 1 *The space* $(\Omega, \mathcal{F}, P)$ *is atom-less.* $\square$

We wish to consider risks that are not essentially bounded. For example, in many branches of insurance it is usually assumed that the amount of claims received during a given time period has a heavy-tail distribution (e.g., a Pareto distribution). If $w(1) < 1$, then the functional (1) satisfies $\rho(X) = +\infty$ for any risk X bounded above but unbounded from below, no matter how thin is the negative tail of the distribution. This is questionable from the economic point of view: the fact that a risk can have catastrophic consequences does not make it automatically unacceptable. It may be perfectly acceptable provided the probability of a catastrophic event is suitably small and is balanced by sufficiently large and plausible positive events. This leads us to the next assumption:

Assumption 2 *The measures* ρ_A*,* ρ_M *are functionals of type* (1) *specified by functions* w_A*,* w_M *respectively, with* $w_A(1) = w_M(1) = 1$*.* $\square$

Notice that if $w(1) = 1$, then expression (1) reduces to

$$\rho(X) = \int_{-\infty}^{0} w(F_X(t))\, dt + \int_{0}^{+\infty} \left(w(F_X(t)) - 1\right)\, dt, \tag{2}$$

where F_X is the distribution function of X.

The agent rates any possible trade $Z \in \mathcal{Z}$ by comparing the standing risk with the sum of standing risk and traded risk, net of the transaction's price. In other words, he compares $\rho_A(Y)$ with the value

$$\phi_Y(Z) = \rho_A(Y + Z - \rho_M(-Z)) = \rho_A(Y + Z) + \rho_M(-Z).$$

Thus, we consider the following optimization problem

Problem 1 *Find $\hat{Z} \in \mathcal{Z}$ such that*

$$\rho_A(Y + \hat{Z}) + \rho_M(-\hat{Z}) = \min\{\rho_A(Y + Z) + \rho_M(-Z) : Z \in \mathcal{Z}\}. \square$$

Obviously, the problem is not fully specified until a definition of the set $\mathcal{Z}$ of tradable risks is provided, and various sets $\mathcal{Z}$ make meaningful versions of Problem 1. For example, consider the following two cases:

Case 1: (insurance)

Suppose Y represents the loss resulting from some insurable event. To simplify, we assume that $\Pr\{Y \leq 0\} = 1$ (i.e., no gain can be made from the insurable event). Considering the agent is seeking to purchase insurance against the loss Y, we set

$$\mathcal{Z} = \{X \in \mathcal{X} : \Pr\{0 \leq X \leq -Y\} = 1\},$$

i.e., we assume that no insurer will refund in excess of the loss incurred and no other payment besides the premium will be requested from the insured.

Case 2: (unconstrained trade)

Suppose Y represents the value at some future time of the portfolio currently detained by a trader. Assuming the trader is free to buy or sell any type of asset, $\mathcal{Z}$ is the set of all risks $Z \in \mathcal{X}$ that can be rated, i.e., such that $\rho_A(Y + Z) + \rho_M(-Z)$ is well defined.

From the examples above we see that in general, there is some relationship between the definition of $\mathcal{Z}$ and the standing risk Y.

The next assumption is quite natural:

Assumption 3 *The standing risk can be rated by both the agent and the market, and none of them rates it as infinitely valuable. Put in other way:*

$$\rho_A(\max(0, Y)) > -\infty, \qquad \text{and} \qquad \rho_M(\max(0, Y)) > -\infty. \square$$

This assumption does not exclude the cases $\rho_A(Y) = +\infty$ or $\rho_M(Y) = +\infty$. The situation $\rho_A(Y) = +\infty$ means that the agent is forced to trade in order to escape a totally unacceptable situation, while the situation $\rho_M(Y) = +\infty$ means that it is not possible for the agent to transfer all the standing risk to the market.

Our last assumption concerns the set of tradable risks:

Assumption 4 *The set $\mathcal{Z}$ contains all the random variables $Z \in \mathcal{X}$ satisfying*

$$\Pr\left\{\min(0,-Y) \le Z \le \max(0,-Y)\right\} = 1. \ \square$$

This means that no regulation or technical obstacle forbids or forces the agent to cede any part of the standing risk. Of course, this does not mean that the agent might be interested in every such transaction. For example, the case $\rho_M(Y) = +\infty$ guarantees the agent will not cede the totality of the standing risk because that would require payment of an infinite premium, though $-Y \in \mathcal{Z}$.

The purpose of this paper is to present and prove the following result:

Theorem 2 *(Under assumptions 1 to 4)*
Let $\theta : \mathbb{R} \mapsto [0,1]$ denote a Lebesgue-measurable function satisfying

$$\begin{aligned} \theta(t) = 1, &\qquad \text{if } w_A\left(F_Y(t)\right) > w_M\left(F_Y(t)\right); \\ \theta(t) = 0, &\qquad \text{if } w_A\left(F_Y(t)\right) < w_M\left(F_Y(t)\right), \end{aligned}$$

and let

$$Z(t) = -\int_0^t \theta(\tau)\, d\tau.$$

$Z(Y)$ is a solution for Problem 1. $\square$

Notice that the solution is not unique. Indeed, if Z is a solution, then translation invariance of the risk measures implies that $Z + c$ is also optimal, provided it is an element of $\mathcal{Z}$. Further, if the set $\{t \in \mathbb{R} : w_A\left(F_Y(t)\right) = w_M\left(F_Y(t)\right)\}$ is a set of positive Lebesgue measure, then there are infinitely many solutions of the form indicated in the Theorem.

The solutions given in Theorem 2 are non-negative for events where the standing risk takes positive values and are non-positive when Y is negative. Thus, they always consist of ceding some risk to the market. The risk is split into infinitesimal layers and the decision on wether to cede a given layer is taken by comparing the values of w_A and w_M, with indifference when they are equal.

3 Example

For example, suppose that

$$w_A(t) = \begin{cases} 4t, & \text{for } t \in [0, \frac{1}{10}); \\ \frac{1+2t}{3}, & \text{for } t \in (\frac{1}{10}, 1], \end{cases} \qquad w_M(t) = \sqrt{t}.$$

Then, we have

$$\begin{aligned} w_A(t) < w_M(t) &\qquad \text{if } t < \frac{1}{16} \text{ or } t > \frac{1}{4}, \\ w_A(t) > w_M(t) &\qquad \text{if } \frac{1}{16} < t < \frac{1}{4}. \end{aligned}$$

Hence, the agent's optimal strategy is to cede all the risk layers between quantiles $\frac{1}{16}$ and $\frac{1}{4}$. To be more concrete, consider the following cases:

Insurable losses

If the random variable Y represents the amount of some insurable loss, then the solution above means that the agent should buy an insurance policy of the type "stop-loss with ceiling". More precisely, a policy that refunds the losses according to the formula

$$Z(y) = \begin{cases} 0, & \text{if } \; y \geq \mathrm{VaR}_Y(\frac{1}{4}); \\ \mathrm{VaR}_Y(\frac{1}{4}) - y, & \text{if } \; \mathrm{VaR}_Y(\frac{1}{16}) < y < \mathrm{VaR}_Y(\frac{1}{4}); \\ \mathrm{VaR}_Y(\frac{1}{4}) - \mathrm{VaR}_Y(\frac{1}{16}), & \text{if } \; y \leq \mathrm{VaR}_Y(\frac{1}{16}), \end{cases}$$

where $\mathrm{VaR}_Y(\alpha) = \sup\{v : \Pr\{Y < v\} \leq \alpha\}$ is the Value at Risk for the loss Y.

Asset return

Suppose that Y represents the return of some asset S at a future time T

$$Y = \frac{S_T - S_0}{S_0}.$$

Then, the solution above means the owner should write an european put and buy an european call on that asset. The strike of the put should be

$$\sup\left\{v : \Pr\{S_T < v\} \leq \frac{1}{16}\right\},$$

and the strike of the call should be

$$\sup\left\{v : \Pr\{S_T < v\} \leq \frac{1}{4}\right\}.$$

The example above is easy to generalize. Provided the graphs of w_A and w_M cross only a finite number of times, the optimal strategy in the asset return problem is always a finite combination of put and call options. In the insurance problem, the solution is a combination of stop-loss treaties with ceiling, possibly with an infinite-ceiling stop-loss.

4 Proof of Theorem 2

The remaining of this paper consists of the proof of Theorem 2.

Notice that, in the case $w_A = w_M = Id$, the theorem is trivial. Also, if $w_A \neq Id = w_M$, then the optimal strategy for the agent is to sell the standing risk for its expected value, while for $w_A = Id \neq w_M$, $Z \equiv 0$ (no trade) is optimal. Thus, we can take the additional assumption:

Assumption 5 *$w_A \neq Id$ and $w_M \neq Id$.* □

All the proofs and results below are conditional on Assumptions 1 to 5, but we will not mention these any more to avoid repetition.

In order to prove Theorem 2, we will proceed by steps. Thus, this Section is organized into subsections as follows:

In Section 4.1, we reformulate the problem as the minimization of a functional with domain in a space of probability laws. We discuss some properties of such functionals and prove existence of a solution for certain sets of tradable risks.

In Section 4.2, we introduce a discretized version of Problem 1 and show that it approximates the original problem when the mesh size goes to zero.

Section 4.3 contains a solution for the discretized problem, under some additional assumptions.

Sections 4.4 and 4.5 contain some technical results about Kusuoka representations of risk measures and absolutely continuous functions, respectively. These results are used in Section 4.6 to show that the solution for the discretized problem approximates a solution of the Problem 1 and the supplementary assumptions introduced in Section 4.3 can be lifted.

4.1 Spaces of joint probabilities

Since the measures ρ_A, ρ_M are law invariant, we are only concerned with joint distributions of pairs (Y, Z), of standing and traded risks.

Since $(\Omega, \mathcal{F}, P)$ is atom-less, the space of all joint probability laws of pairs $(X_1, X_2) \in \mathcal{X} \times \mathcal{X}$ coincides with the set $\mathcal{M}$ of all Borel probability measures $\eta : \mathcal{B}_{\mathbb{R}^2} \mapsto [0, 1]$. Given two random variables $X_1, X_2 \in \mathcal{X}$, we have $X_2 = g(X_1)$ for some function $g : \mathbb{R} \mapsto \mathbb{R}$ if and only if $\Pr\{(X_1, X_2) \in \mathrm{Graph}(g)\} = 1$.

For any $\eta \in \mathcal{M}$, we set

$$\rho_A(\eta) = \rho_A(X_1 + X_2), \qquad \rho_M(\eta) = \rho_M(-X_2),$$

where $X_1, X_2 \in \mathcal{X}$ are any random variables with joint probability law η.

Given a standing risk Y and a set of tradable risks $\mathcal{Z}$, we denote by $\mathcal{M}_{(Y,\mathcal{Z})}$ the set of all measures $\eta \in \mathcal{M}$ that are joint probability laws of (Y, Z) for some $Z \in \mathcal{Z}$. With this notation, the Problem 1 can be reformulated as

Problem 2 *Find $\hat{\eta} \in \mathcal{M}_{(Y,\mathcal{Z})}$ such that*

$$\rho_A(\hat{\eta}) + \rho_M(\hat{\eta}) = \min\left\{\rho_A(\eta) + \rho_M(\eta) : \eta \in \mathcal{M}_{(Y,\mathcal{Z})}\right\}. \ \square$$

For each $k \in [0, +\infty)$, consider the set

$$\mathcal{Z}_k = \{Z \in \mathcal{X} : \Pr\{\min(0, -Y) - k \leq Z \leq \max(0, -Y) + k\} = 1\},$$

and denote the set $\mathcal{M}_{(Y,\mathcal{Z}_k)}$ by $\mathcal{M}_{Y,k}$. Using this notation, the Assumption 4 is equivalent to any of the two inclusions

$$\mathcal{Z}_0 \subset \mathcal{Z}, \qquad \text{or} \qquad \mathcal{M}_{Y,0} \subset \mathcal{M}_{(Y,\mathcal{Z})}.$$

We provide the set $\mathcal{M}$ with the topology of weak convergence over the space C_c, of all continuous functions $g : \mathbb{R}^2 \mapsto \mathbb{R}$ with compact support. This means that a sequence $\{\eta_n \in \mathcal{M}\}_{n\in\mathbb{N}}$ is said to converge to $\eta \in \mathcal{M}$ if and only if

$$\lim \int_{\mathbb{R}^2} g d\eta_n = \int_{\mathbb{R}^2} g d\eta$$

For every $g \in C_c$.

The following propositions state important properties of $\mathcal{M}_{Y,k}$

Proposition 1 *$\mathcal{M}_{Y,k}$ is a compact convex subset of $\mathcal{M}$.* $\square$

Proof. Convexity follows immediately from the fact that convex combinations of probability measures are again probability measures with support contained in the union of the supports of the original measures.

To prove that $\mathcal{M}_{Y,k}$ is compact, we introduce the short notation

$$\langle \eta, g \rangle = \int_{\mathbb{R}^2} g d\eta, \qquad \eta \in \mathcal{M},\ g \in C_c.$$

The set C_c provided with the topology of uniform convergence admits a countable dense subset $\{g_n\}_{n\in\mathbb{N}}$. Since $|\langle \eta, g \rangle| \leq \max_{x\in\mathbb{R}^2} |g(x)| < +\infty$ holds for every $\eta \in \mathcal{M}$, $g \in C_c$, it follows that every sequence $\{\langle \eta_n, g \rangle\}_{n\in\mathbb{N}}$ is a real bounded sequence and therefore contains a convergent subsequence. Thus, we can pick $\left\{\eta_{n_i^1}\right\}_{i\in\mathbb{N}}$, a subsequence of $\{\eta_n \in \mathcal{M}_{Y,k}\}_{n\in\mathbb{N}}$ such that $\left\{\left\langle \eta_{n_i^1}, g_1 \right\rangle\right\}_{i\in\mathbb{N}}$ is convergent. Repeating the same argument, for each $j \in \mathbb{N}$ we can pick $\left\{\eta_{n_i^{j+1}}\right\}_{i\in\mathbb{N}}$, a subsequence of $\left\{\eta_{n_i^j}\right\}_{i\in\mathbb{N}}$ such that $\left\{\left\langle \eta_{n_i^{j+1}}, g_{j+1} \right\rangle\right\}_{i\in\mathbb{N}}$ converges. It follows that $\left\{\eta_{n_i^i}\right\}_{i\in\mathbb{N}}$ is a subsequence of $\{\eta_n\}_{n\in\mathbb{N}}$ such that all the sequences

$$\left\{\left\langle \eta_{n_i^i}, g_j \right\rangle\right\}_{i\in\mathbb{N}}, \qquad j \in \mathbb{N}$$

converge. To see that all the sequences

$$\left\{\left\langle \eta_{n_i^i}, g \right\rangle\right\}_{i\in\mathbb{N}}, \qquad g \in C_c$$

converge, notice that

$$\begin{aligned} &\left|\left\langle \eta_{n_i^i}, g \right\rangle - \left\langle \eta_{n_m^m}, g \right\rangle\right| \leq \\ \leq &\left|\left\langle \eta_{n_i^i}, g_j \right\rangle - \left\langle \eta_{n_m^m}, g_j \right\rangle\right| + \left|\left\langle \eta_{n_i^i}, g - g_j \right\rangle\right| + \left|\left\langle \eta_{n_m^m}, g - g_j \right\rangle\right| \leq \\ \leq &\left|\left\langle \eta_{n_i^i}, g_j \right\rangle - \left\langle \eta_{n_m^m}, g_j \right\rangle\right| + 2 \max_{x\in\mathbb{R}^2} |g(x) - g_j(x)|. \end{aligned}$$

Since $\{g_j\}_{j\in\mathbb{N}}$ is dense in C_c, we see that $\left\{\left\langle \eta_{n_i^i}, g \right\rangle\right\}_{i\in\mathbb{N}}$ is a Cauchy sequence and therefore it is convergent.

This shows that the map $g \mapsto \lim \left\langle \eta_{n_i^i}, g \right\rangle$ is a well defined positive bounded linear functional in C_c. Thus, the Riesz representation theorem (see, e.g. Rudin (1987)) states that there is one unique regular positive measure η satisfying

$$\langle \eta, g \rangle = \lim \left\langle \eta_{n_i^i}, g \right\rangle, \qquad \forall g \in C_c.$$

Let $K = \{(y,z) : \min(0,-y) - k \leq z \leq \max(0,-y) + k\}$. Since $\left\langle \eta_{n_i^i}, g \right\rangle = 0$ holds whenever $\mathrm{Supp}(g) \cap K = \emptyset$, it is clear that $\eta\left(\mathbb{R}^2 \setminus K\right) = 0$. Thus, in order to show that $\eta \in \mathcal{M}_{Y,k}$, we only need to show that $\eta(A \times \mathbb{R}) = \Pr\{Y \in A\}$ holds for every open set $A \subset \mathbb{R}$.

Fix an open set $A \subset \mathbb{R}$, a small $\varepsilon > 0$, and pick a compact set $B \subset A$ such that $\Pr\{Y \in A \setminus B\} < \varepsilon$. There is a function $g \in C_c$ such that

$$\chi_{\{(y,z): y \in B, 0 \leq z \leq y\}} \leq g \leq \chi_{A \times \mathbb{R}}.$$

Then,

$$\lim \eta_{n_i^i}(A \times \mathbb{R}) \geq \lim \left\langle \eta_{n_i^i}, g \right\rangle = \langle \eta, g \rangle.$$

By taking a sequence g_j converging monotonically from below to $\chi_{A \times \mathbb{R}}$, we see that $\lim \eta_{n_i^i}(A \times \mathbb{R}) \geq \eta(A \times \mathbb{R})$. Also, $\lim \eta_{n_i^i}(A \times \mathbb{R}) \leq \lim \left\langle \eta_{n_i^i}, g \right\rangle + \varepsilon = \langle \eta, g \rangle + \varepsilon \leq \eta(A \times \mathbb{R}) + \varepsilon$. Making ε go to zero, we see that $\lim \eta_{n_i^i}(A \times \mathbb{R}) \leq \eta(A \times \mathbb{R})$. ■

Proposition 2 *(For fixed but arbitrary $k \in [0, +\infty)$)*
$\rho_A(\eta) + \rho_M(\eta)$ is well defined for every $\eta \in \mathcal{M}_{Y,k}$. The map $\eta \mapsto \rho_A(\eta) + \rho_M(\eta)$ is lower semicontinuous in $\mathcal{M}_{Y,k}$. □

Proof. Let (Y, Z) have joint probability law $\eta \in \mathcal{M}_{Y,k}$. Then,

$$F_{Y+Z}(t) \geq F_Y(t-k), \quad F_{-Z}(t) \geq F_Y(t-k), \qquad \forall t \geq k. \tag{3}$$

Therefore,

$$\begin{aligned}
\int_0^{+\infty} (w_A(F_{Y+Z}(t)) - 1)\, dt \geq & -k + \int_0^{+\infty} (w_A(F_Y(t)) - 1)\, dt = \\
=& \rho_A(\max(0,Y)) - k; \\
\int_0^{+\infty} (w_M(F_{-Z}(t)) - 1)\, dt \geq & -k + \int_0^{+\infty} (w_M(F_Y(t)) - 1)\, dt = \\
=& \rho_M(\max(0,Y)) - k.
\end{aligned}$$

Hence, Assumption 3 guarantees that $\rho_A(\eta) + \rho_M(\eta)$ is a well defined number in $(-\infty, +\infty]$.

Pick a sequence $\{\eta_n \in \mathcal{M}_{Y,k}\}_{n \in \mathbb{N}}$, converging to $\eta \in \mathcal{M}_{Y,k}$. The sequences $F_{Y+Z_n}(t)$, $F_{-Z_n}(t)$ converge pointwise to $F_{Y+Z}(t)$, $F_{-Z}(t)$, respectively.

The inequalities (3) and Lebesgue's dominated convergence theorem guarantee that

$$\lim \int_0^{+\infty} (w_A(F_{Y+Z_n}(t)) - 1)\, dt = \int_0^{+\infty} (w_A(F_{Y+Z}(t)) - 1)\, dt,$$
$$\lim \int_0^{+\infty} (w_A(F_{-Z_n}(t)) - 1)\, dt = \int_0^{+\infty} (w_A(F_{-Z}(t)) - 1)\, dt,$$

while Fatou's Lemma guarantees that

$$\liminf \int_{-\infty}^{0} w_A(F_{Y+Z_n}(t))dt \geq \int_{-\infty}^{0} w_A(F_{Y+Z}(t))dt,$$
$$\liminf \int_{-\infty}^{0} w_A(F_{-Z_n}(t))dt \geq \int_{-\infty}^{0} w_A(F_{-Z}(t))dt.$$

This proves lower semicontinuity of $\eta \mapsto \rho_A(\eta) + \rho_M(\eta)$. ■

Propositions 1 and 2 have the following obvious corollary:

Corollary 1 *The functional $\eta \mapsto \rho_A(\eta) + \rho_M(\eta)$ admits a minimum in $\mathcal{M}_{Y,k}$, i.e., the Problem 1 admits a solution in $\mathcal{Z}_k$.* □

4.2 Discretization

Problem 1 is easy to approximate by discrete problems of the same type, and we use this feature to characterize its solutions.

First, consider a discretization of the standing risk, i.e., for each $n \in \mathbb{N}$ we consider the discrete random variable Y_n defined as

$$Y_n = \frac{i}{2^n}, \qquad \text{if and only if } Y \in \left(\frac{i-1}{2^n}, \frac{i}{2^n}\right], \quad i \in \mathbb{Z}.$$

The set $\mathcal{X}$ has the obvious discretization $\mathcal{X}^{(n)}$, consisting of all $\mathcal{F}$-measurable random variables taking values in the set $\left\{\frac{i}{2^n}, i \in \mathbb{Z}\right\}$. This introduces the discretization of $\mathcal{Z}_k$

$$\mathcal{Z}_k^{(n)} = \left\{Z \in \mathcal{X}^{(n)} : \Pr\{\min(0, -Y_n) - k \leq Z \leq \max(0, -Y_n) + k\} = 1\right\},$$

corresponding to the discretization of $\mathcal{M}_{Y,k}$, $\mathcal{M}_{Y,k}^{(n)} = \mathcal{M}_{(Y_n, \mathcal{Z}_k^{(n)})}$. Notice that $\mathcal{M}_{Y,k}^{(n)}$ is the set of all $\eta \in \mathcal{M}$ satisfying the following conditions:

(i) η is concentrated in the set

$$\left\{\left(\frac{i}{2^n}, \frac{j}{2^n}\right) : i, j \in \mathbb{Z},\ \min(0, -i) - k2^n \leq j \leq \max(0, -i) + k2^n\right\};$$

(ii) $\sum_{-\infty < i \leq t} \sum_{j \in \mathbb{Z}} \eta\left\{\left(\frac{i}{2^n}, \frac{j}{2^n}\right)\right\} = F_Y\left(\frac{t}{2^n}\right), \qquad \forall t \in \mathbb{Z}.$

The following proposition guarantees that $\mathcal{M}_{Y,k}^{(n)}$ is a "good" discretization of $\mathcal{M}_{Y,k}$.

Proposition 3 *$\rho_A(\eta) + \rho_M(\eta)$ is well defined for every $\eta \in \mathcal{M}_{Y,k}^{(n)}$. Further:*

(i) If $\left\{\rho_n \in \mathcal{M}_{Y,k}^{(n)}\right\}_{n\in\mathbb{N}}$ is a sequence converging to $\eta \in \mathcal{M}_{Y,k}$, then

$$\liminf\left(\rho_A(\eta_n) + \rho_M(\eta_n)\right) \geq \rho_A(\eta) + \rho_M(\eta);$$

(ii) For every $\eta \in \mathcal{M}_{Y,k}$ there exists a sequence $\left\{\rho_n \in \mathcal{M}_{Y,k}^{(n)}\right\}_{n\in\mathbb{N}}$ converging to η such that $\lim\left(\rho_A(\eta_n) + \rho_M(\eta_n)\right) = \rho_A(\eta) + \rho_M(\eta)$. $\square$

Proof. For every $\eta \in \mathcal{M}_{Y,k}^{(n)}$, we have $F_{Y_n}(t) \geq F_Y\left(t - \frac{1}{2^n}\right)$. Hence, the argument used to prove Proposition 2 also proves that $\rho_A(\eta) + \rho_M(\eta)$ is well defined and (i) holds.

Fix $\eta \in \mathcal{M}_{Y,k}$. In order to prove (ii), we only need to find a sequence $\left\{\rho_n \in \mathcal{M}_{Y,k}^{(n)}\right\}_{n\in\mathbb{N}}$ converging to η such that

$$\begin{aligned}
\lim \int_{-\infty}^{0} w_A\left(F_{Y_n+Z_n}(t)\right) dt &\leq \int_{-\infty}^{0} w_A\left(F_{Y+Z}(t)\right) dt, \\
\lim \int_{-\infty}^{0} w_M\left(F_{-Z_n}(t)\right) dt &\leq \int_{-\infty}^{0} w_M\left(F_{-Z}(t)\right) dt.
\end{aligned}$$

Pick the sequence defined as

$$\eta_n\left\{\left(\frac{i}{2^n}, \frac{j}{2^n}\right)\right\} = \eta\left(\left(\frac{i-1}{2^n}, \frac{i}{2^n}\right] \times \left(\frac{j-1}{2^n}, \frac{j}{2^n}\right]\right), \qquad i, j \in \mathbb{Z}.$$

It can be checked that

$$F_{Y_n+Z_n}(t) \leq F_{Y+Z}(t), \quad F_{-Z_n}(t) \leq F_{-Z}\left(t + \frac{1}{2^n}\right), \qquad \forall t \in \mathbb{R}.$$

Hence

$$\begin{aligned}
\int_{-\infty}^{0} w_A\left(F_{Y_n+Z_n}(t)\right) dt &\leq \int_{-\infty}^{0} w_A\left(F_{Y+Z}(t)\right) dt, \\
\int_{-\infty}^{0} w_M\left(F_{-Z_n}(t)\right) dt &\leq \frac{1}{2^n} + \int_{-\infty}^{0} w_M\left(F_{-Z}(t)\right) dt
\end{aligned}$$

for every $n \in \mathbb{N}$, and the result follows. $\blacksquare$

4.3 Solutions for the discretized problem

In this section we solve the discretized problem

Problem 3 *Find $Z_n \in \mathcal{Z}_k^{(n)}$ such that*

$$\rho_A(Y_n + Z_n) + \rho_M(Z_n) = \min\left\{\rho_A(Y_n + Z) + \rho_M(Z) : Z \in \mathcal{Z}_k^{(n)}\right\}.$$

Equivalently, find $\eta_n \in \mathcal{M}_{Y,k}^{(n)}$ such that

$$\rho_A(\eta_n) + \rho_M(\eta_n) = \min\left\{\rho_A(\eta) + \rho_M(\eta) : \eta \in \mathcal{M}_{Y,k}^{(n)}\right\}. \ \Box$$

Proposition 4 *For each $n \in \mathbb{N}$, let J_n denote a function such that*

$$\begin{aligned}
&J_n(0) = 0;\\
&J_n\left(\frac{i+1}{2^n}\right) = J_n\left(\frac{i}{2^n}\right) - \frac{1}{2^n}, \qquad \text{if } w_A\left(F_Y(\frac{i}{2^n})\right) > w_M\left(F_Y(\frac{i}{2^n})\right);\\
&J_n\left(\frac{i+1}{2^n}\right) = J_n\left(\frac{i}{2^n}\right), \qquad \text{if } w_A\left(F_Y(\frac{i}{2^n})\right) < w_M\left(F_Y(\frac{i}{2^n})\right).
\end{aligned}$$

If w_A and w_M are strictly concave, then $Z_n = J_n(Y_n)$ is a solution for Problem 3. $\Box$

In order to prove Proposition 4, we use some intermediate Lemmas. First, we introduce some notation.

Fix $Z_n \in \mathcal{Z}_k^{(n)}$, a solution of Problem 3, and let $\eta \in \mathcal{M}_{Y,k}^{(n)}$ denote the joint probability law of (Y_n, Z_n). For $i \in \mathbb{Z}$, and sufficiently small $\varepsilon \in \mathbb{R}$, let

$$\begin{aligned}
\Delta_A(i,\varepsilon) &= w_A\left(F_{Y_n+Z_n}(\frac{i}{2^n}) + \varepsilon\right) - w_A\left(F_{Y_n+Z_n}(\frac{i}{2^n})\right),\\
\Delta_M(i,\varepsilon) &= w_M\left(F_{-Z_n}(\frac{i}{2^n}) + \varepsilon\right) - w_M\left(F_{-Z_n}(\frac{i}{2^n})\right).
\end{aligned}$$

Let

$$p_{i,j} = \eta\left\{\left(\frac{i}{2^n}, \frac{j}{2^n}\right)\right\}, \qquad i,j \in \mathbb{Z}.$$

Fix $(i_1, j_1) \in \mathbb{Z}^2$ such that $p_{i_1,j_1} > 0$. For $j_2 \neq j_1$, $\varepsilon \in (0, p_{i_1,j_1}]$, let $\tilde{\eta}$ denote the measure corresponding to

$$\tilde{p}_{i,j} = \begin{cases} p_{i,j}, & \text{for } (i,j) \notin \{(i_1,j_1),(i_1,j_2)\},\\ p_{i_1,j_1} - \varepsilon, & \text{for } (i,j) = (i_1,j_1),\\ p_{i_1,j_2} + \varepsilon, & \text{for } (i,j) = (i_1,j_2).\end{cases}$$

Lemma 1 *For $j_1 < j_2$:*

$$\rho_A(\tilde{\eta}) - \rho_A(\eta) = \sum_{s=j_1}^{j_2-1} \frac{\Delta_A(i_1+s,-\varepsilon)}{2^n}, \qquad \rho_M(\tilde{\eta}) - \rho_M(\eta) = \sum_{s=-j_2}^{-j_1-1} \frac{\Delta_M(s,\varepsilon)}{2^n};$$

For $j_1 > j_2$:

$$\rho_A(\tilde{\eta}) - \rho_A(\eta) = \sum_{s=j_2}^{j_1-1} \frac{\Delta_A(i_1+s,\varepsilon)}{2^n}, \qquad \rho_M(\tilde{\eta}) - \rho_M(\eta) = \sum_{s=-j_1}^{-j_2-1} \frac{\Delta_M(s,-\varepsilon)}{2^n}. \ \square$$

Proof. Let $(Y_n, \tilde{Z})$ have joint probability law $\tilde{\eta}$.

For $j_1 < j_2$, we have:

$$F_{Y_n+\tilde{Z}}(t) = \begin{cases} F_{Y_n+Z_n}(t) - \varepsilon, & \text{for } \frac{i_1+j_1}{2^n} \le t < \frac{i_1+j_2}{2^n}, \\ F_{Y_n+Z_n}(t), & \text{otherwise.} \end{cases}$$

$$F_{-\tilde{Z}}(t) = \begin{cases} F_{-Z_n}(t) + \varepsilon, & \text{for } -\frac{j_2}{2^n} \le t < -\frac{j_1}{2^n}, \\ F_{-Z_n}(t), & \text{otherwise.} \end{cases}$$

For $j_1 > j_2$, we have:

$$F_{Y_n+\tilde{Z}}(t) = \begin{cases} F_{Y_n+Z_n}(t) + \varepsilon, & \text{for } \frac{i_1+j_2}{2^n} \le t < \frac{i_1+j_1}{2^n}, \\ F_{Y_n+Z_n}(t), & \text{otherwise.} \end{cases}$$

$$F_{-\tilde{Z}}(t) = \begin{cases} F_{-Z_n}(t) - \varepsilon, & \text{for } -\frac{j_1}{2^n} \le t < -\frac{j_2}{2^n}, \\ F_{-Z_n}(t), & \text{otherwise.} \end{cases}$$

Since

$$\rho_A(\eta) = \sum_{-\infty<i<0} \frac{w_A\left(F_{Y_n+Z_n}(\frac{i}{2^n})\right)}{2^n} + \sum_{0\le i<+\infty} \frac{w_A\left(F_{Y_n+Z_n}(\frac{i}{2^n})\right) - 1}{2^n},$$

$$\rho_M(\eta) = \sum_{-\infty<i<0} \frac{w_M\left(F_{-Z_n}(\frac{i}{2^n})\right)}{2^n} + \sum_{0\le i<+\infty} \frac{w_M\left(F_{-Z_n}(\frac{i}{2^n})\right) - 1}{2^n},$$

the Lemma follows by simple computation. ■

Lemma 2 *For each $i \in \mathbb{Z}$ there exists at most one j such that $p_{i,j} > 0$.* $\square$

Proof. Fix $i_1 \in \mathbb{Z}$ and suppose there are $j_1 \neq j_2$ such that $p_{i_1,j_1} > 0$, $p_{i_1,j_2} > 0$.

Without loss of generality, we may assume that $j_1 < j_2$. Optimality of Z_n, together with Lemma 1 implies that, for every sufficiently small $\varepsilon > 0$:

$$\sum_{s=j_1}^{j_2-1} \Delta_M(-s-1,\varepsilon) \ge \sum_{s=j_1}^{j_2-1} -\Delta_A(i_1+s,-\varepsilon),$$

$$\sum_{s=j_1}^{j_2-1} \Delta_A(i_1+s,\varepsilon) \ge \sum_{s=j_1}^{j_2-1} -\Delta_M(-s-1,-\varepsilon).$$

But, strict concavity of w_A, w_M implies

$$\sum_{s=j_1}^{j_2-1} \Delta_M(-s-1,\varepsilon) < \sum_{s=j_1}^{j_2-1} -\Delta_M(-s-1,-\varepsilon),$$
$$\sum_{s=j_1}^{j_2-1} \Delta_A(i_1+s,\varepsilon) < \sum_{s=j_1}^{j_2-1} -\Delta_A(i_1+s,-\varepsilon).$$

Since this is a contradiction, we see that j_1 must be equal to j_2. ■

Lemma 3 *For any $(i_1, j_1), (i_2, j_2) \in \mathbb{Z}^2$ such that $i_1 < i_2$, $p_{i_1,j_1} > 0$, $p_{i_2,j_2} > 0$, we have:*

$$j_2 \leq j_1 \leq j_2 + i_2 - i_1. \ \square$$

Proof. Fix $(i_1, j_1), (i_2, j_2)$ as above and suppose that $j_1 < j_2$. For $\varepsilon \in (0, \min(p_{i_1,j_1}, p_{i_2,j_2})]$, let $\tilde{\eta} \in \mathcal{M}_{Y,k}^{(n)}$ denote the measure corresponding to

$$\tilde{p}_{i,j} = \begin{cases} p_{i,j}, & \text{for } (i,j) \notin \{(i_1,j_1),(i_1,j_2),(i_2,j_1),(i_2,j_2)\}, \\ p_{i_1,j_1} - \varepsilon, & \text{for } (i,j) = (i_1,j_1), \\ p_{i_1,j_2} + \varepsilon, & \text{for } (i,j) = (i_1,j_2), \\ p_{i_2,j_1} + \varepsilon, & \text{for } (i,j) = (i_2,j_1), \\ p_{i_2,j_2} - \varepsilon, & \text{for } (i,j) = (i_2,j_2). \end{cases}$$

It is clear that $F_{-\tilde{Z}} \equiv F_{-Z_n}$. Hence,

$$\rho_A(\tilde{\eta}) + \rho_M(\tilde{\eta}) - (\rho_A(\eta) + \rho_M(\eta)) = \rho_A(\tilde{\eta}) - \rho_A(\eta).$$

Using Lemma 1, one obtains

$$\rho_A(\tilde{\eta}) - \rho_A(\eta) = \sum_{s=j_1}^{j_2-1} \frac{\Delta_A(i_2+s,\varepsilon) + \Delta_A(i_1+s,-\varepsilon)}{2^n}.$$

Therefore, strict concavity of w_A implies $\rho_A(\tilde{\eta}) - \rho_A(\eta) < 0$, a contradiction to the optimality of Z_n.

Now, suppose that $i_1 + j_1 > i_2 + j_2$. Let $d = i_1 + j_1 - (i_2 + j_2)$, and let $\tilde{\eta} \in \mathcal{M}_{Y,k}^{(n)}$ denote the measure corresponding to

$$\tilde{p}_{i,j} = \begin{cases} p_{i,j}, & \text{for } (i,j) \notin \{(i_1,j_1),(i_1,j_2),(i_2,j_1-d),(i_2,j_2+d)\}, \\ p_{i_1,j_1} - \varepsilon, & \text{for } (i,j) = (i_1,j_1), \\ p_{i_1,j_1-d} + \varepsilon, & \text{for } (i,j) = (i_1,j_1-d), \\ p_{i_2,j_2} - \varepsilon, & \text{for } (i,j) = (i_2,j_2), \\ p_{i_2,j_2+d} + \varepsilon, & \text{for } (i,j) = (i_2,j_2+d). \end{cases}$$

In this case, one obtains

$$\rho_A(\tilde{\eta}) + \rho_M(\tilde{\eta}) - (\rho_A(\eta) + \rho_M(\eta)) = \rho_M(\tilde{\eta}) - \rho_M(\eta) =$$
$$= \sum_{s=1}^{d} \frac{\Delta_M(1 - j_1 - s, -\varepsilon) + \Delta_M(-j_2 - s, \varepsilon)}{2^n},$$

and strict concavity of w_M implies that this sum is again strictly negative. ■

Proof of Proposition 4. Lemmas 2 and 3 show that there is a sequence $\{j_i, \in \mathbb{Z}\}_{i\in\mathbb{Z}}$ such that

$$p_{i,j} > 0 \quad \text{only if} \quad j = j_i,$$

and this sequence satisfies $j_{i+1} \in \{j_i, j_i - 1\},\ \forall i \in \mathbb{Z}$.

Fix $i_1 \in \mathbb{Z}$, and assume that $j_{i_1+1} = j_{i_1} - 1$. Consider the sequence

$$\tilde{j}_i = \begin{cases} j_i, & \text{if } i \le i_1, \\ j_i + 1, & \text{if } i > i_1, \end{cases}$$

and let $\tilde{\eta} \in \mathcal{M}^{(n)}_{Y,k}$ denote the measure corresponding to

$$\tilde{p}_{i,j} = \begin{cases} p_{i,j}, & \text{if } i \le i_1, \\ p_{i,j+1}, & \text{if } i > i_1. \end{cases}$$

In other words, we consider the risk

$$\tilde{Z}(Y_n) = \begin{cases} Z_n(Y_n), & \text{for } Y_n \le \frac{i_1}{2^n}; \\ Z_n(Y_n) + \frac{1}{2^n}, & \text{for } Y_n > \frac{i_1}{2^n}. \end{cases}$$

It can be checked that

$$F_{Y_n+\tilde{Z}}(\frac{i}{2^n}) = \begin{cases} F_{Y_n+Z_n}(\frac{i}{2^n}), & \text{for } i < i_1 + j_{i_1}, \\ F_Y(\frac{i_1}{2^n}), & \text{for } i = i_1 + j_{i_1}, \\ F_{Y_n+Z_n}(\frac{i-1}{2^n}), & \text{for } i > i_1 + j_{i_1}; \end{cases}$$
$$F_{-\tilde{Z}}(\frac{i}{2^n}) = \begin{cases} F_{-Z_n}(\frac{i}{2^n}), & \text{for } i < -j_{i_1}, \\ F_{-Z_n}(\frac{i+1}{2^n}), & \text{for } i \ge -j_{i_1}. \end{cases}$$

Then, a simple but tedious computation leads to

$$w_A(\tilde{\eta}) - w_A(\eta) = \frac{w_A\left(F_Y(\frac{i_1}{2^n})\right) - 1}{2^n},$$
$$w_M(\tilde{\eta}) - w_M(\eta) = \frac{1 - w_M\left(F_{-Z_n}(\frac{-j_{i_1}}{2^n})\right)}{2^n}.$$

But, due to Lemma 3 and $j_{i_1+1} = j_{i_1} - 1$, we have

$$F_{-Z_n}(\frac{-j_{i_1}}{2^n}) = \Pr\{Z_n \geq \frac{j_{i_1}}{2^n}\} = F_Y(\frac{i_1}{2^n}).$$

Hence,

$$w_A(\tilde{\eta}) + w_M(\tilde{\eta}) - (w_A(\eta) + w_M(\eta)) = \frac{w_A\left(F_Y(\frac{i_1}{2^n})\right) - w_M\left(F_Y(\frac{i_1}{2^n})\right)}{2^n}.$$

Hence, optimality of Z_n requires $w_A\left(F_Y(\frac{i_1}{2^n})\right) \geq w_M\left(F_Y(\frac{i_1}{2^n})\right)$, and in the case $w_A\left(F_Y(\frac{i_1}{2^n})\right) = w_M\left(F_Y(\frac{i_1}{2^n})\right)$, both Z_n and $\tilde{Z}$ are optimal.

Assuming that $j_{i_1+1} = j_{i_1}$, and considering the risk

$$\tilde{Z}(Y_n) = \begin{cases} Z_n(Y_n), & \text{for } Y_n \leq \frac{i_1}{2^n}; \\ Z_n(Y_n) - \frac{1}{2^n}, & \text{for } Y_n > \frac{i_1}{2^n}, \end{cases}$$

then, the same argument shows that optimality of Z_n requires $w_A\left(F_Y(\frac{i_1}{2^n})\right) \leq w_M\left(F_Y(\frac{i_1}{2^n})\right)$, and Z_n and $\tilde{Z}$ are both optimal if $w_A\left(F_Y(\frac{i_1}{2^n})\right) = w_M\left(F_Y(\frac{i_1}{2^n})\right)$. ■

4.4 Concave functions

Definition 3 *Let $\mathcal{W}_Y$ denote the set of all continuous concave functions $w : [0,1] \mapsto [0,1]$ such that*

$$w(0) = 0, \qquad w(1) = 1, \qquad \int_0^{+\infty} (1 - w(F_Y(t)))\, dt < +\infty,$$

and let $\mathcal{W}_Y^+$ denote the set of all $w \in \mathcal{W}_Y$ that are strictly concave. □

Notice that Assumption 3 is equivalent to

$$w_A, w_M \in \mathcal{W}_Y.$$

Lemma 4 *There is a continuous strictly convex strictly decreasing function $\alpha : [0,1] \mapsto [0,1]$ such that*

$$\alpha(1) = 0, \qquad \int_0^{+\infty} \alpha(F_Y(t)) dt < +\infty. \ \square$$

Proof. Let G denote a concave continuous strictly increasing function such that

$$G(0) = 0, \qquad G(t) \leq F_Y(t),\ \forall t \geq 0, \qquad \lim_{t \to +\infty} G(t) = 1$$

(for example, G may be piecewise linear), and let

$$\alpha(t) = \frac{1}{\left(1 + G^{-1}(t)\right)^2}, \qquad t \in [0,1).$$

α can be extended by continuity to $t = 1$ and it satisfies the conditions in the Lemma ■

Lemma 5 *For any $w \in \mathcal{W}_Y \setminus \{Id\}$, there is $u \in \mathcal{W}_Y^+$ such that*

$$u(t) < w(t), \qquad \forall t \in (0,1). \ \square$$

Proof. Since w is concave and is not the identity, there exists $t_1 \in (0,1)$ such that $w(t_1) > t_1$. Therefore, the graph of w lies above the graph of

$$w_1(t) = \begin{cases} \frac{w(t_1)}{t_1} t, & \text{for } t \in [0, t_1]; \\ w(t_1) + \frac{1-w(t_1)}{1-t_1}(t - t_1), & \text{for } t \in (t_1, 1]. \end{cases}$$

For any $\varepsilon \in (0,1)$, the function $u_\varepsilon(t) = t + \varepsilon t(1-t)$ is continuous, strictly concave, strictly increasing in $[0,1]$, with $u_\varepsilon(0) = 0$, $u_\varepsilon(1) = 1$. For sufficiently small $\varepsilon > 0$, its graph lies below the graph of w_1 and hence $u_\varepsilon(t) < w(t)$, $\forall t \in (0,1)$. If $E(Y) < +\infty$, then $u_\varepsilon \in \mathcal{W}_Y^+$.

In order to prove the Lemma for the case $E(Y) = +\infty$, fix a function α satisfying Lemma 4. The function $w_1 = w - \alpha$ is continuous and strictly concave. Therefore, it admits left-derivative at every point of $(0,1]$, and $w_1'(t) < \frac{w_1(t)}{t}$ for every $t \in (0,1)$. Hence, for $\theta < 1$ sufficiently close to 1 and $\varepsilon > 0$ sufficiently small, the function

$$u(t) = \begin{cases} \frac{w_1(\theta)}{\theta} t + \varepsilon t(\theta - t), & \text{if } t \in [0,\theta]; \\ w_1(t), & \text{if } t \in (\theta, 1], \end{cases}$$

satisfies the Lemma. ■

Lemma 6 *For any $w \in \mathcal{W}_Y \setminus \{Id\}$ and any open set $A \subset [0,1]$, there are $w_1, w_2 \in \mathcal{W}_Y^+$ such that $w_1 \le w_2 \le w$ and $w_1(t) < w_2(t)$ if and only if $t \in A$.* $\square$

Proof. Due to Lemma 5, we can pick $u_1, u_2 \in \mathcal{W}_Y^+$ such that

$$u_1(t) < u_2(t) < w(t), \qquad \forall t \in (0,1).$$

First, consider the case $A = (a,b)$, for some $0 \le a < b \le 1$, such that the graph of u_1 lies below the straight line interpolating the points $(a, u_2(a))$, $(b, u_2(b)$. Let

$$u_{(a,b)}(t) = \begin{cases} u_2(t), & \text{for } t \in [0,1] \setminus (a,b); \\ u_2(a) + \frac{u_2(b) - u_2(a)}{b-a}(t-a), & \text{for } t \in (a,b). \end{cases}$$

Since u_2 is strictly concave and $u_{(a,b)} \ge u_1$, we see the Lemma holds with $w_1 = (1-2\lambda)w + \lambda u_{(a,b)} + \lambda u_2$, $w_2 = (1-2\lambda)w + 2\lambda u_2$, for any $\lambda \in (0,1)$.

Now, consider an arbitrary open set $A \subset (0,1)$, and pick u_1, u_2 as above. For any $t \in (0,1)$ there are $a < t$, $b > t$ such that the graph of u_1 lies below the straight line interpolating the points $(a, u_2(a))$, $(b, u_2(b)$. Hence, A is the union of countably many intervals $\{(a_i, b_i)\}_{i \in \mathbb{N}}$ with this property. For each of these intervals, pick the function $u_{(a_i,b_i)}$ as above. Then, we see the Lemma holds with

$$w_1 = (1-2\lambda)w + \lambda \sum_{i=1}^{\infty} \frac{1}{2^i} u_{(a_i,b_i)} + \lambda u_2, \qquad w_2 = (1-2\lambda)w + 2\lambda u_2,$$

with $\lambda \in (0,1)$. ■

4.5 Approximation of absolutely continuous functions

Lemma 7 *Fix $\theta : \mathbb{R} \mapsto \mathbb{R}$, a measurable function such that*

$$0 \le \theta(t) \le 1, \qquad a.e.\ t \in \mathbb{R}.$$

There exists a sequence $\{\theta_n : \mathbb{R} \mapsto \mathbb{R}\}_{n\in\mathbb{N}}$ such that:

(i) *For every $n \in \mathbb{N}$, $\theta_n(\mathbb{R}) = \{0,1\}$, and the support of θ_n is contained in the support of θ;*

(ii) *The sequence $Z_n(t) = \int_0^t \theta_n(\tau)d\tau$, $t \in \mathbb{R}$ converges uniformly towards $Z(t) = \int_0^t \theta(\tau)d\tau$.* □

Proof. Consider the function

$$u(t) = \begin{cases} 1, & \text{if } \theta(t) > 0; \\ 0, & \text{if } \theta(t) = 0. \end{cases}$$

We define θ_n recursively for each interval $\left(\frac{i}{n}, \frac{i+1}{n}\right]$:

$$\theta_n(t) = \begin{cases} u(t)\ \forall t \in \left(\frac{i}{n}, \frac{i+1}{n}\right], & \text{if } Z(\frac{i}{n}) > Z_n(\frac{i}{n}); \\ 0\ \forall t \in \left(\frac{i}{n}, \frac{i+1}{n}\right], & \text{if } Z(\frac{i}{n}) \le Z_n(\frac{i}{n}) \end{cases} \qquad \text{for } i \ge 0;$$

$$\theta_n(t) = \begin{cases} u(t)\ \forall t \in \left(\frac{i}{n}, \frac{i+1}{n}\right], & \text{if } Z(\frac{i+1}{n}) < Z_n(\frac{i+1}{n}); \\ 0\ \forall t \in \left(\frac{i}{n}, \frac{i+1}{n}\right], & \text{if } Z(\frac{i+1}{n}) \ge Z_n(\frac{i+1}{n}) \end{cases} \qquad \text{for } i < 0.$$

Then, $|Z_n(t) - Z(t)| \le \frac{1}{n}$ for every $t \in \mathbb{R}$. ■

4.6 The continuous problem

In this section we use the results in Sections 4.1 to 4.5 to prove Theorem 2 in its full generality.

We start with the following particular case

Proposition 5 *Suppose $\mathcal{Z} = \mathcal{Z}_k$ and let*

$$\theta(t) = \begin{cases} 1, & \text{if } w_A(F_Y(t)) > w_M(F_Y(t)), \\ 0, & \text{if } w_A(F_Y(t)) \le w_M(F_Y(t)), \end{cases}$$

$$Z(t) = -\int_0^t \theta(\tau)d\tau.$$

If $w_A, w_M \in \mathcal{W}_Y^+$, then $Z(Y)$ is a solution for the Problem 1. □

Proof. Let $\left\{Z_n \in \mathcal{Z}_k^{(n)}\right\}_{n\in\mathbb{N}}$ denote the sequence of solutions for the discretized problems, described in Proposition 4 with

$$\begin{aligned} J_n\left(\frac{i+1}{2^n}\right) &= J_n\left(\frac{i}{2^n}\right) - \frac{1}{2^n}, & \text{if } w_A\left(F_Y(\frac{i}{2^n})\right) > w_M\left(F_Y(\frac{i}{2^n})\right), \\ J_n\left(\frac{i+1}{2^n}\right) &= J_n\left(\frac{i}{2^n}\right), & \text{if } w_A\left(F_Y(\frac{i}{2^n})\right) \le w_M\left(F_Y(\frac{i}{2^n})\right). \end{aligned}$$

We identify the function $i \mapsto J_n\left(\frac{i}{2^n}\right)$ with the piecewise linear function $y \mapsto Z_n(y)$ that interpolates the points $\left(\frac{i}{2^n}, J_n\left(\frac{i}{2^n}\right)\right)$, $i \in \mathbb{Z}$.

The sequence of functions $y \mapsto Z_n(y)$ converges to the function $y \mapsto Z(y)$, uniformly over compact intervals. Therefore, Proposition 3 guarantees optimality of Z. ■

To extend Proposition 5 to the case when w_A, w_M are concave but not strictly concave, we use the following Lemma:

Lemma 8 *Fix $w \in \mathcal{W}_Y$ and sequences $\{w_A^{(n)} \in \mathcal{W}_Y\}_{n\in\mathbb{N}}$, $\{w_M^{(n)} \in \mathcal{W}_Y\}_{n\in\mathbb{N}}$ such that*

$$\begin{aligned} &w \le w_A^{(n)} \le w_A^{(n+1)} \le w_A, \quad w \le w_M^{(n)} \le w_M^{(n+1)} \le w_M, \qquad \forall n \in \mathbb{N}, \\ &\lim w_A^{(n)}(t) = w_A(t), \quad \lim w_M^{(n)}(t) = w_M(t), \qquad \forall t \in [0,1]. \end{aligned}$$

Let $\rho_A^{(n)}$, $\rho_M^{(n)}$ denote the risk measures corresponding to the functions $w_A^{(n)}$, $w_M^{(n)}$, respectively. Consider a sequence $\{\eta_n \in \mathcal{M}_{Y,k}\}_{n\in\mathbb{N}}$ such that

$$\rho_A^{(n)}(\eta_n) + \rho_M^{(n)}(\eta_n) = \min\left\{\rho_A^{(n)}(\eta) + \rho_M^{(n)}(\eta) : \eta \in \mathcal{M}_{Y,k}\right\}.$$

If η_n converges to $\eta \in \mathcal{M}_{Y,k}$, then η is a solution for Problem 2. □

Proof. Fix $\hat{\eta} \in \mathcal{M}_{Y,k}$, a solution for Problem 2. Using Lebesgue's monotone convergence and dominated convergence theorems, one obtains

$$\rho_A(\hat{\eta}) + \rho_M(\hat{\eta}) = \lim \rho_A^{(n)}(\hat{\eta}) + \rho_M^{(n)}(\hat{\eta}).$$

Thus, optimality of η_n implies

$$\rho_A(\hat{\eta}) + \rho_M(\hat{\eta}) \ge \limsup \rho_A^{(n)}(\eta_n) + \rho_M^{(n)}(\eta_n).$$

Using again Lebesgue's monotone convergence and dominated convergence theorems, one obtains

$$\rho_A(\hat{\eta}) + \rho_M(\hat{\eta}) \ge \rho_A(\eta) + \rho_M(\eta),$$

hence η is optimal. ■

Now, consider the case $w_A, w_M \in \mathcal{W}_Y$.

Proposition 6 *Fix an open set $V \subset (0,1)$ and consider the functions*

$$\theta(t) = \begin{cases} 1, & \textit{if} \quad w_A(F_Y(t)) > w_M(F_Y(t)) \textit{or} \\ & \quad (w_A(F_Y(t)) = w_M(F_Y(t)) \textit{ and } F_Y(t) \in V), \\ 0, & \textit{otherwise}, \end{cases}$$
$$Z(t) = -\int_0^t \theta(\tau)d\tau.$$

$Z(Y)$ is a solution for the Problem 1. □

Proof. Using Lemma 6, pick $w_1, w_2 \in \mathcal{W}_Y^+$ such that $w_1 \leq w_2 \leq \min(w_A, w_M)$ and $w_1(t) < w_2(t)$ if and only if $t \in V$. Let $\{w_A^{(n)} \in \mathcal{W}_Y^+\}_{n\in\mathbb{N}}$, $\{w_M^{(n)} \in \mathcal{W}_Y^+\}_{n\in\mathbb{N}}$ be the sequences:

$$w_A^{(n)} = \frac{n-2}{n} w_A + \frac{1}{n} w_1 + \frac{1}{n} w_2,$$
$$w_M^{(n)} = \frac{n-2}{n} w_M + \frac{2}{n} w_1.$$

Consider the functions

$$\theta_n(t) = \begin{cases} 1, & \text{if } w_A^{(n)}(F_Y(t)) > w_M^{(n)}(F_Y(t)), \\ 0, & \text{if } w_A^{(n)}(F_Y(t)) \leq w_M^{(n)}(F_Y(t)), \end{cases}$$
$$Z_n(t) = -\int_0^t \theta_n(\tau)d\tau.$$

Proposition 5 guarantees that each $Z_n(Y)$ is a solution for the Problem 1 with risk measures $\rho_A^{(n)}$, $\rho_M^{(n)}$, corresponding to the functions $w_A^{(n)}$, $w_M^{(n)}$, respectively.

Since Z_n converges to Z uniformly on compact intervals, the result follows from Lemma 8. ■

Due to Lemma 7, every function Z of the type described in the Theorem 2 can be approximated by a sequence of functions of the type described in Proposition 6. Taking into account lower semicontinuity of $\eta \mapsto \rho_A(\eta) + \rho_M(\eta)$, this concludes the proof of Theorem 2 for the case $\mathcal{Z} = \mathcal{Z}_k$.

To conclude the proof of Theorem 2, consider an arbitrary $\mathcal{Z}$ such that $\mathcal{Z}_0 \subset \mathcal{Z}$. The argument used to prove Lemma 2 shows that for any $Z \in \mathcal{Z}$ there exists a $\tilde{Z}$ lying in some $\mathcal{Z}_k$ (k large enough) such that $\rho_A(Y + \tilde{Z}) + \rho_M(-\tilde{Z}) < \rho_A(Y + Z) + \rho_M(-Z)$. Due to translation-invariance, $\rho_A(Y + \tilde{Z} - \tilde{Z}(0)) + \rho_M(-\tilde{Z} + \tilde{Z}(0)) = \rho_A(Y + \tilde{Z}) + \rho_M(-\tilde{Z})$. Hence, Problem 1 always admits a particular solution lying in $\mathcal{Z}_0$.

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