Existence and uniqueness of SDE solutions

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We will study the question of existence and uniqueness of solutions to a non-linear stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \quad t \in [t_0, T]$$

with initial value $x(t_0) = x_0$, where $0 \leq t_0 < T < \infty$.

Some of the main (mathematical) questions regarding such equations:

- Is there a solution?
- If there is a solution, is it unique?
- What kind of properties do solutions have?
- How can solutions be obtained in practice?
Background and notation

- Let 
  - \((\Omega, \mathcal{F}, P)\) be a probability space.
  - \(B(t) = (B_1(t), \ldots, B_m(t))^T\) be an \(m\)-dimensional Brownian motion.
  - \(x_0\) be an \(\mathcal{F}_{t_0}\)-measurable (where \(0 \leq t_0 < T < \infty\)) \(\mathbb{R}^d\)-valued random variable such that \(E|x_0|^2 < \infty\).
  - \(f : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^d\) and \(g : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^{d \times m}\) be Borel measurable.

Consider the \(d\)-dimensional stochastic differential equation of Itô type

\[ dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \quad t_0 \leq t \leq T, \]

with initial value \(x(t_0) = x_0\).

The initial value problem above is equivalent to the following stochastic integral equation

\[ x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)dB(s), \quad t_0 \leq t \leq T. \]
Background and notation

Definition (SDE solution)

We say that the stochastic process \( \{ x(t) \}_{t_0 \leq t \leq T} \) is a solution of the stochastic differential equation

\[
\text{d}x(t) = f(x(t), t)\text{d}t + g(x(t), t)\text{d}B(t)
\]

with initial condition \( x(t_0) = x_0 \) if the following conditions hold:

(i) \( \{ x(t) \} \) is continuous and \( \mathcal{F}_t \)-adapted;
(ii) \( \{ f(x(t), t) \} \in L^1([t_0, T]; \mathbb{R}^d) \) and \( \{ g(x(t), t) \} \in L^2([t_0, T]; \mathbb{R}^{d \times m}) \);
(iii) the integral equation

\[
x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\text{d}s + \int_{t_0}^{t} g(x(s), s)\text{d}B(s)
\]

holds for every \( t \in [t_0, T] \) with probability 1.
A solution \( \{x(t)\} \) is said to be unique if any other solution \( \{\bar{x}(t)\} \) is indistinguishable from \( \{x(t)\} \), that is, almost all their sample paths agree

\[
P\{x(t) = \bar{x}(t) \text{ for all } t_0 \leq t \leq T\} = 1.\]
Example

- Let us consider the the stochastic differential equation given by
  \[ dN_t = rN_t dt + \alpha N_t dB_t. \]

- Equivalently, we have that
  \[ \frac{dN_t}{N_t} = rd t + \alpha dB_t. \]

- Hence
  \[ \int_0^t \frac{dN_s}{N_s} = rt + \alpha B_t \quad (B_0 = 0). \]
Example

To evaluate the integral on the left hand side, we use the Itô formula for the function

\[ g(t, x) = \ln x , \quad x > 0 \]

to obtain

\[
\begin{align*}
\frac{d(\ln N_t)}{N_t} & = \frac{1}{N_t} \cdot dN_t + \frac{1}{2} \left( - \frac{1}{N_t^2} \right) (dN_t)^2 \\
& = \frac{dN_t}{N_t} - \frac{1}{2N_t^2} \cdot \alpha^2 N_t^2 dt = \frac{dN_t}{N_t} - \frac{1}{2} \alpha^2 dt .
\end{align*}
\]

Hence

\[
\frac{dN_t}{N_t} = d(\ln N_t) + \frac{1}{2} \alpha^2 dt .
\]
Example

- Therefore, from \( \int_0^t \frac{dN_s}{N_s} = rt + \alpha B_t \) we conclude that

\[
\ln \frac{N_t}{N_0} = (r - \frac{1}{2} \alpha^2)t + \alpha B_t
\]

or

\[
N_t = N_0 \exp((r - \frac{1}{2} \alpha^2)t + \alpha B_t).
\]

- The solution \( N_t \) is a process of the form

\[
X_t = X_0 \exp(\mu t + \alpha B_t), \quad \mu, \alpha \text{ constants}
\]

- We call such processes \textit{geometric Brownian motion}.
Remark

- It seems reasonable that if \( B_t \) is independent of \( N_0 \) we should have
  \[
  E[N_t] = E[N_0]e^{rt}.
  \]

- To see that this is indeed the case, we let
  \[
  Y_t = e^{\alpha B_t}.
  \]

- Apply Itô’s formula to obtain
  \[
  dY_t = \alpha e^{\alpha B_t} dB_t + \frac{1}{2} \alpha^2 e^{\alpha B_t} dt
  \]
  or
  \[
  Y_t = Y_0 + \alpha \int_0^t e^{\alpha B_s} dB_s + \frac{1}{2} \alpha^2 \int_0^t e^{\alpha B_s} ds.
  \]
Remark

Since $E[\int_0^t e^{\alpha B_s} dB_s] = 0$, we get

$$E[Y_t] = E[Y_0] + \frac{1}{2} \alpha^2 \int_0^t E[Y_s] ds$$

i.e.

$$\frac{d}{dt} E[Y_t] = \frac{1}{2} \alpha^2 E[Y_t], \quad E[Y_0] = 1 .$$

Therefore, we get that

$$E[Y_t] = e^{\frac{1}{2} \alpha^2 t} .$$

We conclude that

$$E[N_t] = E[N_0] e^{rt} .$$
Background and notation

If we take $g(x, t) \equiv 0$, then the SDE above reduces to

$$\dot{x}(t) = f(x(t), t), \quad t \in [t_0, T].$$

Note that the initial condition $x(t_0) = x_0$ may still be a random variable.

Example

Consider the following classical example

$$\dot{x} = 3x^{2/3}, \quad t \in [t_0, T]$$

with initial condition $x(t_0) = 1_A$, where $A \in \mathcal{F}_{t_0}$. It is possible to check that for each $0 < \alpha < T - t_0$, the stochastic process

$$x(t) = x(t, \omega) = \begin{cases} (t - t_0 + 1)^3 & \text{para } t_0 \leq t \leq T, \omega \in A \\ 0 & \text{para } t_0 \leq t \leq t_0 + \alpha, \omega \notin A \\ (t - t_0 - \alpha)^3 & \text{para } t_0 + \alpha < t \leq T, \omega \notin A \end{cases}$$

is a solution of the equation above. This initial value problem has an infinite number of solutions.
Background and notation

Example

Consider yet another simple equation

\[ \dot{x} = x^2, \quad t \in [t_0, T] \]

with initial condition given by \( x(t_0) = x_0 \), a random variable which takes values larger than \( 1/(T - t_0) \).

It is possible to check that the initial value problem above has a unique solution

\[ x(t) = \left( \frac{1}{x_0} - (t - t_0) \right)^{-1} \]

for \( t_0 \leq t < t_0 + 1/x_0 < T \).

However, there is no solution for this initial value problem which is defined for all \( t \in [t_0, T] \).
Existence and uniqueness of solutions

**Theorem (Existence and uniqueness of solution)**

Assume that there exist two positive constants $\overline{K}$ and $K$ such that the following two conditions hold:

(i) **Lipschitz condition**: for all $x, y \in \mathbb{R}^d$ and $t \in [t_0, T]$

$$\max\{|f(x, t) - f(y, t)|^2, |g(x, t) - g(y, t)|^2\} \leq \overline{K}|x - y|^2 ;$$

(ii) **Linear growth condition**: for all $(x, t) \in \mathbb{R}^d \times [t_0, T]$

$$\max\{|f(x, t)|^2, |g(x, t)|^2\} \leq K(1 + |x|^2) .$$

Let $x_0$ be a random variable which is independent of the $\sigma$-algebra $\mathcal{F}^{(m)}_{\infty}$ generated by $B_s(\cdot), \ s \geq 0$ and such that $E|x_0|^2 < \infty$.

Then there exists a unique $t$-continuous solution $X_t(\omega)$ of the initial value problem

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) , \ t_0 \leq t \leq T , \ x(t_0) = x_0$$

with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}^{x_0}_t$ generated by $x_0$ and $B_s(\cdot), \ s \leq t$. Furthermore, such solution belongs to $\mathcal{M}^2([t_0, T]; \mathbb{R}^d)$. 
Existence and uniqueness of solutions

- We start by proving some auxiliary lemmas to prepare for the proof of the theorem above.

**Theorem**

Let \( p \geq 2 \) and let \( g \in \mathcal{M}^2([0, T]; \mathbb{R}^{d \times m}) \) be such that

\[
E \left[ \int_0^T |g(s)|^p \, ds \right] < \infty.
\]

Then

\[
E \left| \int_0^T g(s) \, dB(s) \right|^p \leq \left( \frac{p(p - 1)}{2} \right)^{p/2} T^{(p-2)/2} E \left[ \int_0^T |g(s)|^p \, ds \right].
\]

In particular, the equality holds for \( p = 2 \).
Existence and uniqueness of solutions

Proof.

For $0 \leq t \leq T$, set

$$x(t) = \int_0^t g(s)dB(s).$$

Using Itô’s formula (and Itô’s integral properties), one can obtain

$$E|\textbf{x}(t)|^p = \frac{p}{2} E \int_0^t \left( |\textbf{x}(s)|^{p-2}|g(s)|^2 + (p-2)|\textbf{x}(s)|^{p-4}|\textbf{x}^T(s)g(s)|^2 \right) ds$$

$$\leq \frac{p(p-1)}{2} E \int_0^t |\textbf{x}(s)|^{p-2}|g(s)|^2 ds.$$

Recall the Hölder’s inequality, for $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $X \in L^p$, $Y \in L^q$ we have

$$E|XY| \leq (E|X|^p)\frac{1}{p} (E|Y|^q)\frac{1}{q}$$
Existence and uniqueness of solutions

Proof.

Using the previous inequality one then sees that

\[
E|x(t)|^p \leq \frac{p(p-1)}{2} \left( E \int_0^t |x(s)|^p \, ds \right)^{\frac{p-2}{p}} \left( E \int_0^t |g(s)|^p \, ds \right)^{\frac{2}{p}}
\]

\[
= \frac{p(p-1)}{2} \left( \int_0^t E|x(s)|^p \, ds \right)^{\frac{p-2}{p}} \left( E \int_0^t |g(s)|^p \, ds \right)^{\frac{2}{p}}.
\]

Noting that \( E|x(t)|^p \) is nondecreasing in \( t \), we obtain

\[
E|x(t)|^p \leq \frac{p(p-1)}{2} \left[ tE|x(t)|^p \right]^{\frac{p-2}{p}} \left( E \int_0^t |g(s)|^p \, ds \right)^{\frac{2}{p}}.
\]

This last inequality yields

\[
E|x(t)|^p \leq \left( \frac{p(p-1)}{2} \right)^{\frac{p}{2}} t^{\frac{p-2}{2}} E \int_0^t |g(s)|^p \, ds,
\]

concluding the proof.
Existence and uniqueness of solutions

Theorem

Under the assumptions of the previous theorem,

\[ E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t g(s) dB(s) \right|^p \right] \leq \left( \frac{p^3}{2(p - 1)} \right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E \int_0^T |g(s)|^p ds. \]

Proof.

- Recall that the stochastic integral \( \int_0^t g(s) dB(s) \) is a martingale.
- By the Doob martingale inequality we have that

\[ E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t g(s) dB(s) \right|^p \right] \leq \left( \frac{p}{p - 1} \right)^p E \left| \int_0^T g(s) dB(s) \right|^p. \]

- Using the previous theorem, we then obtain the desired inequality.
Existence and uniqueness of solutions

**Theorem (Gronwall’s inequality)**

Let $T > 0$ and $c \geq 0$. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0, T]$, and let $v(\cdot)$ be a nonnegative integrable function on $[0, T]$. If

$$u(t) \leq c + \int_0^t v(s)u(s)ds, \quad \text{for all } 0 \leq t \leq T,$$

then

$$u(t) \leq c \exp\left(\int_0^t v(s)ds\right), \quad \text{for all } 0 \leq t \leq T.$$

**Proof.**

- Without loss of generality we may assume that $c > 0$.
- Set
  
  $$z(t) = c + \int_0^t v(s)u(s)ds, \quad \text{for } 0 \leq t \leq T.$$

- Then $u(t) \leq z(t)$. 
Existence and uniqueness of solutions

Proof.

Clearly, we have that

$$\log(z(t)) = \log(c) + \int_0^t \frac{v(s)u(s)}{z(s)} \, ds \leq \log(c) + \int_0^t v(s) \, ds.$$ 

This implies

$$z(t) \leq c \exp\left(\int_0^t v(s) \, ds\right), \quad \text{for} \quad 0 \leq t \leq T.$$ 

The required inequality follows since $u(t) \leq z(t)$. 

\[ \square \]
Existence and uniqueness of solutions

Lemma

Assume that the linear growth condition holds. If $x(t)$ is a solution of equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t),$$

then

$$E\left( \sup_{t_0 \leq t \leq T} |x(t)|^2 \right) \leq (1 + 3E|x_0|^2)e^{3K(T-t_0)(T-t_0+4)}.$$

In particular, $x(t)$ belongs to $M^2([t_0, T; \mathbb{R}^d])$.

Proof.

- For every integer $n \geq 1$, define the stopping time

  $$\tau_n = \min\{T, \inf\{t \in [t_0, T] : |x(t)| \geq n\}\}.$$

- Set $x_n(t) = x(\min\{t, \tau_n\})$ for $t \in [t_0, T]$. 
Existence and uniqueness of solutions

Proof.

- Then $x_n(t)$ satisfies the equation

$$x_n(t) = x_0 + \int_{t_0}^{t} f(x_n(s), s) l_{[t_0, \tau_n]}(s) ds + \int_{t_0}^{t} g(x_n(s), s) l_{[t_0, \tau_n]}(s) ds.$$  

- Using the elementary inequality

$$|a + b + c|^2 \leq 3(|a|^2 + |b|^2 + |c|^2),$$

the Hölder inequality and the linear growth condition, one can show that

$$|x_n(t)|^2 \leq 3|x_0|^2 + 3K(t-t_0) \int_{t_0}^{t} (1+|x_n(s)|^2) ds + 3 \left| \int_{t_0}^{t} g(x_n(s), s) l_{[t_0, \tau_n]}(s) ds \right|^2.$$
Existence and uniqueness of solutions

Proof.

Hence, using again the linear growth condition and the previous theorem, we obtain that

\[
E\left( \sup_{t_0 \leq s \leq t} |x_n(s)|^2 \right) \leq 3E|x_0|^2 + 3K(T - t_0) \int_{t_0}^{t} (1 + E|x_n(s)|^2) ds
\]

\[+ 12E \int_{t_0}^{t} |g(x_n(s), s)|^2 I_{[t_0, \tau_n]}(s) ds \]

\[\leq 3E|x_0|^2 + 3K(T - t_0 + 4) \int_{t_0}^{t} (1 + E|x_n(s)|^2) ds.\]

Consequently

\[1 + E\left( \sup_{t_0 \leq s \leq t} |x_n(s)|^2 \right) \]

\[\leq 1 + 3E|x_0|^2 + 3K(T - t_0 + 4) \int_{t_0}^{t} \left[1 + E\left( \sup_{t_0 \leq r \leq s} |x_n(r)|^2 \right)\right] ds.\]
Existence and uniqueness of solutions

Proof.

Now, Gronwall inequality implies that

\[ 1 + E\left( \sup_{t_0 \leq t \leq T} |x_n(t)|^2 \right) \leq (1 + 3E|x_0|^2)e^{3K(T-t_0)(T-t_0+4)}. \]

Thus

\[ E\left( \sup_{t_0 \leq t \leq \tau_n} |x_n(t)|^2 \right) \leq (1 + 3E|x_0|^2)e^{3K(T-t_0)(T-t_0+4)} \]

The required inequality follows by letting \( n \to \infty \).
Existence and uniqueness of solutions

Proof of theorem of existence and uniqueness of solutions.

Uniqueness

Let \( x(t) \) and \( \bar{x}(t) \) be two solutions.

By the previous lemma, both of them belong to \( \mathcal{M}^2([t_0, T]; \mathbb{R}^d) \).

Note that

\[
 x(t) - \bar{x}(t) = \int_{t_0}^{t} f(x(s), s) - f(\bar{x}(s), s) \, ds + \int_{t_0}^{t} g(x(s), s) - g(\bar{x}(s), s) \, dB(s).
\]

Using the Hölder inequality, the previous theorem and Lipschitz condition, one can show (in the same way as in the proof of the previous lemma) that

\[
 E \left( \sup_{t_0 \leq s \leq t} |x(s) - \bar{x}(s)|^2 \right) \leq 2\bar{K}(T + 4) \int_{t_0}^{t} E \left( \sup_{t_0 \leq r \leq s} |x(r) - \bar{x}(r)|^2 \right) \, ds.
\]
Existence and uniqueness of solutions

Proof.

- The Gronwall inequality then yields that

\[ E \left( \sup_{t_0 \leq t \leq T} |x(t) - \overline{x}(t)|^2 \right) = 0. \]

- Hence, \( x(t) = \overline{x}(t) \) for all \( t_0 \leq t \leq T \) almost surely, concluding the proof of uniqueness of solutions.
Existence and uniqueness of solutions

Proof.

**Existence**

- Set \( x_0(t) \equiv x_0 \) and, for \( n = 1, 2, \ldots \), define the Picard iterations

\[
x_n(t) = x_0 + \int_{t_0}^{t} f(x_{n-1}(s), s) ds + \int_{t_0}^{t} g(x_{n-1}(s), s) dB(s)
\]

for \( t \in [t_0, T] \).
  - Note that \( x(\cdot) \in M^2([t_0, T]; \mathbb{R}^d) \).
- It is easy to see by induction that \( x_n(\cdot) \in M^2([t_0, T]; \mathbb{R}^d) \), because we have that

\[
E|x_n(t)|^2 \leq c_1 + 3K(T + 1) \int_{t_0}^{t} E|x_{n-1}(s)|^2 ds
\]

where \( c_1 = 3E|x_0|^2 + 3KT(T + 1) \).
Existence and uniqueness of solutions

Proof.

For any $k \geq 1$

$$\max_{1 \leq n \leq k} E|x_n(t)|^2 \leq c_1 + 3K(T + 1) \int_{t_0}^{t} \max_{1 \leq n \leq k} E|x_{n-1}(s)|^2 ds$$

$$\leq c_1 + 3K(T + 1) \int_{t_0}^{t} \left( E|x_0|^2 + \max_{1 \leq n \leq k} E|x_n(s)|^2 \right) ds$$

$$\leq c_2 + 3K(T + 1) \int_{t_0}^{t} \max_{1 \leq n \leq k} E|x_n(s)|^2,$$

where $c_2 = c_1 + 3KT(T + 1)E|x_0|^2$.

Gronwall inequality implies that

$$\max_{1 \leq n \leq k} E|x_n(t)|^2 \leq c_2 e^{3KT(T+1)}.$$

Since $k$ is arbitrary, we must have

$$E|x_n(t)|^2 \leq c_2 e^{3KT(T+1)} \text{ for all } t_0 \leq t \leq T, n \geq 1.$$
Existence and uniqueness of solutions

Proof.

Note that

$$|x_1(t) - x_0(t)|^2 = |x_1(t) - x_0|^2 \leq 2 \left| \int_{t_0}^{t} f(x_0, s) \, ds \right|^2 + 2 \left| \int_{t_0}^{t} g(x_0, s) \, dB(s) \right|^2.$$ 

Taking the expectation and using the linear growth condition we get

$$E|x_1(t) - x_0(t)|^2 \leq 2K(t - t_0)^2(1 + E|x_0|^2) + 2K(t - t_0)(1 + E|x_0|^2) \leq C,$$

where $C = 2K(T - t_0 + 1)(T - t_0)(1 + E|x_0|^2)$.

We now claim that for $n \geq 0$,

$$E|x_{n+1}(t) - x_n(t)|^2 \leq \frac{C[M(t - t_0)]^n}{n!}, \quad \text{for } t_0 \leq t \leq T,$$

where $M = 2\bar{K}(T - t_0 + 1)$. 

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Proof.

- By induction, we shall show that $E|x_{n+1}(t) - x_n(t)|^2 \leq \frac{C[M(t-t_0)]^n}{n!}$ still holds for $n + 1$.
- Note that

$$|x_{n+2}(t) - x_{n+1}(t)|^2 \leq 2 \left| \int_{t_0}^{t} [f(x_{n+1}(s), s) - f(x_n(s), s)] ds \right|^2$$

$$+ 2 \left| \int_{t_0}^{t} [g(x_{n+1}(s), s) - g(x_n(s), s)] dB(s) \right|^2.$$

- Taking the expectation and using the Lipschitz condition we derive that

$$E|x_{n+2}(t) - x_{n+1}(t)|^2 \leq 2\overline{K}(T - t_0 + 1)E \int_{t_0}^{t} |x_{n+1}(s) - x_n(s)|^2 ds$$

$$\leq M \int_{t_0}^{t} E|x_{n+1}(s) - x_n(s)|^2 ds$$

$$\leq M \int_{t_0}^{t} \frac{C[M(s-t_0)]^n}{n!} ds = \frac{C[M(t-t_0)]^{n+1}}{(n+1)!}.$$

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Proof.

Furthermore, replacing \( n \) with \( n - 1 \) we see that

\[
\sup_{t_0 \leq t \leq T} |x_{n+1}(t) - x_n(t)|^2 \leq 2\overline{K}(T - t_0) \int_{t_0}^{T} |x_n(s) - x_{n-1}(s)|^2 ds
\]

\[
+ 2 \sup_{t_0 \leq t \leq T} \left| \int_{t_0}^{T} [g(x_n(s), s) - g(x_{n-1}(s), s)] dB(s) \right|^2.
\]

Taking the expectation and using the previous theorem, we find that

\[
E\left( \sup_{t_0 \leq t \leq T} |x_{n+1}(t) - x_n(t)|^2 \right) \leq 2\overline{K}(T - t_0 + 4) \int_{t_0}^{T} E|x_n(s) - x_{n-1}(s)|^2 ds
\]

\[
\leq 4M \int_{t_0}^{T} \frac{C[M(s - t_0)]^{n-1}}{(n - 1)!} ds
\]

\[
= \frac{4C[M(T - t_0)]^n}{n!}.
\]
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Proof.

Hence

$$P\left\{ \sup_{0 \leq t \leq T} |x_{n+1}(t) - x_n(t)| > \frac{1}{2^n} \right\} \leq \frac{4C[4M(T-t_0)]^n}{n!}.$$ 

Since $$\sum_{n=0}^{\infty} \frac{4C[4M(T-t_0)]^n}{n!} < \infty$$, the Borel-Cantelli lemma yields that for almost all $$\omega \in \Omega$$ there exists a positive integer $$n_0 = n_0(\omega)$$ such that

$$\sup_{0 \leq t \leq T} |x_{n+1}(t) - x_n(t)| \leq \frac{1}{2^n}, \quad n \geq n_0.$$ 

It follows that, with probability 1, the partial sums

$$x_0(t) + \sum_{i=0}^{n-1} [x_{i+1}(t) - x_i(t)] = x_n(t)$$

are convergent uniformly in $$t \in [0, T]$$.
Existence and uniqueness of solutions

Proof.

Denote the limit by \( x(t) \).

- Clearly, \( x(t) \) is continuous and \( \mathcal{F}_t \)-adapted.
- For every \( t \), \( \{x_n(t)_{n \geq 1}\} \) is a Cauchy sequence in \( L^2 \).
- Hence \( x_n(t) \to x(t) \) in \( L^2 \).

Letting \( n \to \infty \) in

\[
E| x_n(t) |^2 \leq c_2 e^{3KT(T+1)}
\]

gives

\[
E| x(t) |^2 \leq c_2 e^{3KT(T+1)}, \quad \text{for all } t_0 \leq t \leq T.
\]

Therefore \( x(\cdot) \in \mathcal{M}^2([t_0, T]; \mathbb{R}^d) \).

It remains to show that \( x(t) \) satisfies equation

\[
x(t) = \int_{t_0}^{t} f(x(s), s) ds + \int_{t_0}^{t} g(x(s), s) dB(s).
\]
Existence and uniqueness of solutions

Proof.

Note that

\[
E \left| \int_{t_0}^{t} f(x_n(s), s)ds - \int_{t_0}^{t} f(x(s), s)ds \right|
\]
\[+ E \left| \int_{t_0}^{t} g(x_n(s), s)dB(s) - \int_{t_0}^{t} g(x(s), s)dB(s) \right|^2 \]
\[\leq \overline{K}(T - t_0 + 1) \int_{t_0}^{T} E|x_n(s) - x(s)|^2ds \rightarrow 0
\]

Hence we can let \( n \rightarrow \infty \) in

\[
x_n(t) = x_0 + \int_{t_0}^{t} f(x_{n-1}(s), s)ds + \int_{t_0}^{t} g(x_{n-1}(s), s)dB(s)
\]
Existence and uniqueness of solutions

Proof.

We obtain that

\[ x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)dB(s), \quad \text{on } t_0 \leq t \leq T \]

as desired.

In the proof above we show that the Picard iterations \( x_n(t) \) converge to the unique solution \( x(t) \) of the equation

\[ dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \]