An Overview of the Martingale Representation Theorem

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Definition

Denote by $\{\mathcal{F}_t\}_{t \geq 0}$ the filtration generated by the one-dimensional Brownian motion $B_t$ and by $\mathcal{B}$ the Borel $\sigma$-algebra on $[0, \infty)$. Let $\mathcal{V} = \mathcal{V}(S, T)$ be the class of functions $f : [0, \infty) \times \Omega \to \mathbb{R}$ such that

(i) $(t, \omega) \to f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable.

(ii) $f(t, \omega)$ is $\mathcal{F}_t$-adapted.

(iii) $E \left[ \int_S^T (f(t, \omega))^2 \, dt \right] < \infty$. 

Definition (The Itô integral)

Let $f \in \mathcal{V}(S, T)$. Then the Itô integral of $f$ (from $S$ to $T$) is defined by

$$
\int_S^T f(t, \omega) dB_t(\omega) = \lim_{n \to \infty} \int_S^T \phi_n(t, \omega) dB_t(\omega) \quad \text{limit in } L^2(P),
$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$
E \left[ \int_S^T (f(t, \omega) - \phi_n(t, \omega))^2 dt \right] \to 0 \quad \text{as } n \to \infty,
$$

where the limit in (1) exists and does not depend on the choice of $\{\phi_n\}$, as long as (2) holds.
Background and notation

Properties of the Itô integral:
- Itô isometry:
  \[ E \left[ \left( \int_{S}^{T} f(t, \omega) dB_t(\omega) \right)^2 \right] = E \left[ \int_{S}^{T} (f(t,\omega))^2 \, dt \right] \text{ for all } f \in \mathcal{V}(S, T) \]
- Linearity
  \[ E \left[ \int_{S}^{T} f dB_t \right] = 0 \]
- \[ \int_{S}^{T} f dB_t \text{ is } \mathcal{F}_T \text{-measurable} \]
- Existence of a continuous version
- Martingale property

Definition (Martingale)

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(X = \{X_t : t \geq 0\}\) a stochastic process on it.
We say that \(X_t\) is a martingale with respect to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) if
(i) \(X\) is adapted to \(\mathcal{F}_t\).
(ii) \(E[|X_t|] < \infty\) for all \(t\).
(iii) \(E[X_s|\mathcal{F}_t] = X_t\) for all \(s \geq t\).
Background and notation

**Theorem (Itô formula)**

Let $X_t$ be an Itô process given by

$$dX_t = ud_t + vd_{B_t}$$

and let $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$. Then

$$Y_t = g(t, X_t)$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2,$$

where $(dX_t)^2 = (dX_t) . (dX_t)$ is computed according to the “multiplication table”

$$dt . dt = dt . dB_t = dB_t . dt = 0, \quad dB_t dB_t = dt.$$
Martingale Representation Theorem

- Let $B(t) = (B_1(t), \ldots, B_n(t))$ be a $n$-dimensional Brownian motion.
- We know that if $\nu \in \mathcal{V}^n$ then the Itô integral

$$X_t = X_0 + \int_0^t \nu(s, w) dB(s); \quad t \geq 0$$

is always a martingale w.r.t. filtration $\mathcal{F}^{(n)}_t$.
- In this talk we will prove that the converse is also true:
  - any $\mathcal{F}^{(n)}_t$-martingale (w.r.t. $P$) can be represented as an Itô integral.
  - this result is know as martingale representation theorem.
Martingale Representation Theorem

Theorem (The martingale representation theorem)

Let $B(t) = (B_1(t), \ldots, B_n(t))$ be $n$-dimensional. Suppose $M_t$ is an $\mathcal{F}_{t}^{(n)}$-martingale (w.r.t. $P$) and that $M_t \in L^2(P)$ for all $t \geq 0$. Then there exists a unique stochastic process $g(s, \omega)$ such that $g \in \mathcal{V}^{(n)}(0, t)$ for all $t \geq 0$ and

$$M_t(\omega) = E[M_0] + \int_0^t g(s, \omega)dB(s) \quad \text{a.s., for all } t \geq 0.$$

- The key result for the proof of the martingale representation theorem is the Itô representation theorem.
Martingale Representation Theorem

**Theorem (The Itô representation theorem)**

Let $F \in L^2(\mathcal{F}_T^n, P)$. Then there exists a unique stochastic process $f(t, \omega) \in \mathcal{V}^n(0, T)$ such that

$$F(\omega) = E[F] + \int_0^T f(t, \omega) dB(t).$$

- For the proof of the Itô representation theorem we need to prove some auxiliary lemmas.
Martingale Representation Theorem

Lemma (Doob-Dinkyn Lemma)

Let \((\Omega, \Sigma)\) and \((S, A)\) be measurable spaces and \(f : \Omega \to S\) be measurable, i.e., \(f^{-1}(A) \subset \Sigma\). Then a function \(g : \Omega \to \mathbb{R}\) is measurable relative to the \(\sigma\)-algebra \(f^{-1}(A)\) [i.e., \(g^{-1}(B) \subset f^{-1}(A)\)] if and only if there is a measurable function \(h : S \to \mathbb{R}\) such that \(g = h \circ f\).

Proof.

\((\Leftarrow)\)

- Let \(g = h \circ f : \Omega \to \mathbb{R}\) be measurable.
  
- Then
    \[
    g^{-1}(B) = (h \circ f)^{-1}(B) = f^{-1}(h^{-1}(B)) \subset f^{-1}(A)
    \]
  
  since \(h^{-1}(B) \subset A\).
Martingale Representation Theorem

Proof.

(⇒)

- Let $g$ be $f^{-1}(\mathcal{A})$-measurable, where $f^{-1}(\mathcal{A})$ is a $\sigma$-algebra contained in $\Sigma$.
- We start by checking that it is enough to prove the result for simple functions $g = \sum_{i=1}^{n} a_i \chi_{A_i}$, $A_i \in f^{-1}(\mathcal{A})$.

Recall that for a measurable function $g$ w.r.t. the $\sigma$-algebra $f^{-1}(\mathcal{A})$, there exists a sequence of simple functions $g_n$, measurable w.r.t. $f^{-1}(\mathcal{A})$, such that $g_n(\omega) \to g(\omega)$ as $n \to \infty$ for each $\omega \in \Omega$.

- Assuming the result holds for simple functions, there is an $\mathcal{A}$-measurable $h_n : S \to \mathbb{R}$, $g_n = h_n \circ f$, for each $n \geq 1$.
- Define $S_0 = \{s \in S : h_n(s) \to \tilde{h}(s), n \to \infty\}$. Then: $S_0 \in \mathcal{A}$ and $f(\Omega) \subset S$. 

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Proof.

- Let $h(s) = \tilde{h}(s)$ if $s \in S_0$ and $h(s) = 0$ if $s \in S - S_0$
  - Then $h$ is $\mathcal{A}$-measurable and $g(\omega) = h(f(\omega))$, $\omega \in \Omega$, as required

- Assume now that $g = \sum_{i=1}^{n} a_i \chi_{A_i}$, $A_i = f^{-1}(B_i) \in f^{-1}(\mathcal{A})$ for a $B_i \in \mathcal{A}$.
- Define $h = \sum_{i=1}^{n} a_i \chi_{B_i}$
  - Then $h : S \rightarrow \mathbb{R}$ is $\mathcal{A}$-measurable and simple.

- Thus

  $$h(f(\omega)) = \sum_{i=1}^{n} a_i \chi_{B_i}(f(\omega)) = \sum_{i=1}^{n} a_i \chi_{f^{-1}(B_i)}(\omega)$$

  $$= \sum_{i=1}^{n} a_i \chi_{A_i}(\omega) = g(\omega), \quad \omega \in \Omega,$$

  and $h \circ f = g$
Martingale Representation Theorem

- If $S = \mathbb{R}^n$ and $\mathcal{A}$ is the Borel $\sigma$-algebra of $\mathbb{R}^n$, then there is an $h : \mathbb{R}^n \to \mathbb{R}$, Borel measurable, which satisfies the requirements.

Corollary

Let $(\Omega, \Sigma)$ and $(\mathbb{R}^n, \mathcal{A})$ be measurable spaces, and $f : \Omega \to \mathbb{R}^n$ be measurable. Then $g : \Omega \to \mathbb{R}$ is $f^{-1}(\mathcal{A})$-measurable if and only if there is a Borel measurable function $h : \mathbb{R}^n \to \mathbb{R}$ such that $g = h(f_1, f_2, \ldots, f_n) = h \circ f$ where $f = (f_1, \ldots, f_n)$.

Lemma

Fix $T > 0$. The set of random variables

$$\{\phi(B_{t_1}, \ldots, B_{t_n}); \ t_i \in [0, T], \phi \in C_0^\infty(\mathbb{R}^n), \ n = 1, 2, \ldots\}$$

is dense in $L^2(\mathcal{F}_T, P)$. 

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Proof.

- Let \( \{t_i\}_{i=1}^{\infty} \) be a dense subset of \([0, T]\);
- Let \( \mathcal{H}_n \) be the \(\sigma\)-algebra generated by \( B_{t_1}(\cdot), \ldots, B_{t_n}(\cdot) \).
  - Then clearly \( \mathcal{H}_n \subseteq \mathcal{H}_{n+1} \).
  - \( \mathcal{F}_T \) is the smallest \(\sigma\)-algebra containing all the \( \mathcal{H}_n \).
- Choose \( g \in L^2(\mathcal{F}_T, P) \).
  - then by the martingale convergence theorem, we have that
    \[
g = E[g | \mathcal{F}_T] = \lim_{n \to \infty} E[g | \mathcal{H}_n].\]
- The limit is pointwise a.e. \((P)\) and in \(L^2(\mathcal{F}_T, P)\).
  - By the Doob-Dynkin Lemma we can write, for each \( n \),
    \[
    E[g | \mathcal{H}_n] = g_n(B_{t_1}, \ldots, B_{t_n})
    \]
    for some Borel measurable function \( g_n : \mathbb{R}^n \to \mathbb{R} \).
- Each such \( g_n(B_{t_1}, \ldots, B_{t_n}) \) can be approximated in \( L^2(\mathcal{F}_T, P) \) by functions \( \phi_n(B_{t_1}, \ldots, B_{t_n}) \), where \( \phi_n \in C_0^\infty(\mathbb{R}^n) \) and the result follows.
Martingale Representation Theorem

Lemma

The linear span of random variables of the type

\[ \exp \left\{ \int_0^T h(t) dB_t(\omega) - \frac{1}{2} \int_0^T h^2(t) dt \right\}; \quad h \in L^2[0, T] \text{(deterministic)} \]

is dense in $L^2(\mathcal{F}_T, P)$.

Proof.

- Suppose $g \in L^2(\mathcal{F}_T, P)$ is orthogonal (in $L^2(\mathcal{F}_T, P)$) to all functions of the form
  \[ \exp \left\{ \int_0^T h(t) dB_t(\omega) - \frac{1}{2} \int_0^T h^2(t) dt \right\}. \]
  - Then in particular
    \[ G(\lambda) := \int_{\Omega} \exp\{\lambda_1 B_{t_1}(\omega) + \ldots + \lambda_n B_{t_n}(\omega)\} g(\omega) dP(\omega) = 0 \]
    for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and all $t_1, \ldots, t_n \in [0, T]$. 

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Proof.

- The function $G(\lambda)$ is real analytic in $\lambda \in \mathbb{R}^n$ and hence $G$ has an analytic extension to the complex space $\mathbb{C}^n$ given by

  $$G(z) := \int_{\Omega} \exp\{z_1 B_{t_1}(\omega) + \ldots + z_n B_{t_n}(\omega)\} g(\omega) dP(\omega)$$

  for all $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

- Since $G = 0$ on $\mathbb{R}^n$ and $G$ is analytic, $G = 0$ on $\mathbb{C}^n$.
  - In particular, $G(iy_1, \ldots, iy_n) = 0$ for all $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. 


Martingale Representation Theorem

Proof.

But then we get, for $\phi \in C_0^\infty(\mathbb{R}^n)$

$$
\int_{\Omega} \phi(B_{t_1}, \ldots, B_{t_n}) g(\omega) dP(\omega)
$$

$$
= \int_{\Omega} (2\pi)^{-\frac{n}{2}} \left( \int_{\mathbb{R}^n} \hat{\phi}(y) e^{i(y_1 B_{t_1} + \ldots + y_n B_{t_n})} dy \right) g(\omega) dP(\omega)
$$

$$
= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{\phi}(y) \left( \int_{\Omega} e^{i(y_1 B_{t_1} + \ldots + y_n B_{t_n})} g(\omega) dP(\omega) \right) dy
$$

$$
= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{\phi}(y) G(iy) dy = 0,
$$

where

$$
\hat{\phi}(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \phi(x) e^{-ix \cdot y} dx
$$

is the Fourier transform of $\phi$. 
Martingale Representation Theorem

Proof.

- We have used the inverse Fourier transform theorem

$$\phi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{\phi}(y) e^{-ix \cdot y} \, dy.$$ 

- Since

$$\int_{\Omega} \phi(B_{t_1}, \ldots, B_{t_n}) g(\omega) \, dP(\omega) = 0$$

and from the previous lemma $g$ is orthogonal to a dense subset of $L^2(\mathcal{F}_T, P)$, we conclude that $g = 0$.

- Therefore the linear span of the functions

$$\exp \left\{ \int_0^T h(t) \, dB_t(\omega) - \frac{1}{2} \int_0^T h^2(t) \, dt \right\},$$

must be dense in $L^2(\mathcal{F}_T, P)$. 


Martingale Representation Theorem

Proof (Itô representation theorem).

- We consider only the case $n = 1$ (the proof in the general case is similar).
- First assume that $F$ is of the form

$$F(\omega) = \exp \left\{ \int_0^T h(t) dB_t(\omega) - \frac{1}{2} \int_0^T h^2(t) dt \right\}$$

for some $h(t) \in L^2[0, T]$.
- Define

$$Y_t(\omega) = \exp \left\{ \int_0^t h(s) dB_s(\omega) - \frac{1}{2} \int_0^t h^2(s) ds \right\}; \quad 0 \leq t \leq T.$$  

- Then by Itô’s formula

$$dY_t = Y_t(h(t) dB_t - \frac{1}{2} h^2(t) dt) + \frac{1}{2} Y_t(h(t) dB_t)^2 = Y_t h(t) dB_t$$
Martingale Representation Theorem

Proof.

So that

\[ Y_t = 1 + \int_0^t Y_s h(s) dB_s; \quad t \in [0, T]. \]

Therefore

\[ F = Y_T = 1 + \int_0^T Y_s h(s) dB_s \]

\[ E[F] = 1. \]

\[ F(\omega) = E[F] + \int_0^T f(t, \omega) dB(t) \]

holds in this case.

By linearity \( F(\omega) = E[F] + \int_0^T f(t, \omega) dB(t) \) also holds for linear combinations of functions of the form

\[ \exp \left\{ \int_0^T h(t) dB_t(\omega) - \frac{1}{2} \int_0^T h^2(t) dt \right\}. \]
Martingale Representation Theorem

Proof.

- If \( F \in L^2(\mathcal{F}_T, P) \) is arbitrary, we approximate \( F \) in \( L^2(\mathcal{F}_T, P) \) by linear combinations \( F_n \) of functions of the form

\[
\exp \left\{ \int_0^T h(t) dB_t(\omega) - \frac{1}{2} \int_0^T h^2(t) dt \right\}.
\]

- Then for each \( n \) we have

\[
F_n(\omega) = E[F_n] + \int_0^T f_n(s, \omega) dB_s(\omega)
\]

where \( f_n \in \mathcal{V}(0, T) \).
Martingale Representation Theorem

Proof.

By the Itô isometry

\[
E[(F_n - F_m)^2] = E[(E[F_n - F_m] + \int_0^T (f_n - f_m)dB)^2]
\]

\[
= E([F_n - F_m])^2 + \int_0^T E[(f_n - f_m)^2]dt \to 0 \quad n, m \to \infty
\]

- \{f_n\} is a Cauchy sequence in \(L^2([0, T] \times \Omega)\).
- and hence converges to some \(f \in L^2([0, T] \times \Omega)\).
- since \(f_n \in \mathcal{V}(0, T)\) we have \(f \in \mathcal{V}(0, T)\).

Again using the Itô isometry we see that

\[
F = \lim_{n \to \infty} F_n = \lim_{n \to \infty} \left( E[F_n] + \int_0^T f_n dB \right) = E[F] + \int_0^T f dB
\]

- Hence the representation \(F(\omega) = E[F] + \int_0^T f(t, \omega)dB(t)\) holds for all \(F \in L^2(\mathcal{F}_T, P)\).
Martingale Representation Theorem

Proof.

- The uniqueness follows from the Itô isometry.

- Suppose

\[
F(\omega) = E[F] + \int_0^T f_1(t, \omega) dB_t(\omega) = E[F] + \int_0^T f_2(t, \omega) dB_t(\omega)
\]

with \( f_1, f_2 \in \mathcal{V}(0, T) \).

- Then

\[
0 = E[(\int_0^T (f_1(t, \omega) - f_2(t, \omega)) dB_t(\omega))^2] = \int_0^T E[(f_1(t, \omega) - f_2(t, \omega))^2] dt.
\]

- and therefore \( f_1(t, \omega) = f_2(t, \omega) \) for a.a. \( (t, \omega) \in [0, T] \times \Omega \).
Martingale Representation Theorem

Proof (Martingale Representation Theorem).

- $n = 1$.
- By the Itô representation theorem applied to $T = t$ and $F = M_t$, we have that
  - for all $t$ there exists a unique $f(t)(s, \omega) \in L^2(\mathcal{F}_T, P)$ such that
    \[
    M_t(\omega) = E[M_t] + \int_0^t f(t)(s, \omega)dB_s(\omega) = E[M_0] + \int_0^t f(t)(s, \omega)dB_s(\omega).
    \]
- Now assume $0 \leq t_1 \leq t_2$.
- Then
  \[
  M_{t_1} = E[M_{t_2} | \mathcal{F}_{t_1}] = E[M_0] + E \left[ \int_0^{t_2} f(t_2)(s, \omega)dB_s(\omega) | \mathcal{F}_{t_1} \right]
  \]
  \[
  = E[M_0] + \int_0^{t_1} f(t_2)(s, \omega)dB_s(\omega). \quad (3)
  \]
Martingale Representation Theorem

Proof.

- But we also have

\[ M_{t_1} = E[M_0] + \int_0^{t_1} f^{(t_1)}(s, \omega) dB_s(\omega). \]  

(4)

- Hence, comparing (3) and (4) we get that

\[ 0 = E \left[ \left( \int_0^{t_1} (f^{(t_2)} - f^{(t_1)}) d B \right)^2 \right] = \int_0^{t_1} E[(f^{(t_2)} - f^{(t_1)})] ds \]

- Therefore

\[ f^{(t_1)}(s, \omega) = f^{(t_2)}(s, \omega) \]

for a.a. \((s, \omega) \in [0, t_1] \times \Omega.\)
Martingale Representation Theorem

Proof.

- So we can define \( f(s, \omega) \) for a.a. \( s \in [0, \infty) \times \Omega \) by setting \( f(s, \omega) = f^{(N)}(s, \omega) \), if \( s \in [0, N] \).
- Then we get

\[
M_t = E[M_0] + \int_0^t f^{(t)}(s, \omega) \, dB_s(\omega) = E[M_0] + \int_0^t f(s, \omega) \, dB_s(\omega),
\]

for all \( t \geq 0 \).