Notes on Furstenberg’s paper "Noncommuting Random products", Part II

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Theorem (1)

If $M$ is a boundary of $G$, for every absolutely continuous measure, w.r.t. Haar, with compact support $\mu \in \mathcal{P}(G)$ there is a unique measure $\nu \in \mathcal{P}(M)$ such that $\mu \ast \nu = \nu$.

For the existence define $P_\mu : \mathcal{P}(M) \to \mathcal{P}(M), \quad P_\mu(\nu) = \mu \ast \nu$

$M$ compact $\Rightarrow \mathcal{P}(M)$ is weakly compact
$\Rightarrow P_\mu$ has fixed points
Stochastic Matrices

Let $P = [P_{i,j}] \in M_{d \times d}(\mathbb{R})$.

**Definition**
We say that $P$ is a *stochastic matrix* iff $P_{i,j} \geq 0$ and $\sum_{j=1}^{d} P_{i,j} = 1$.

We write $P^n = [P_{i,j}^{(n)}]$.

**Definition**
A state $i = 1, \ldots, d$ is called *transient* iff $P_{i,i}^{(n)} = 0 \ \forall \ n \in \mathbb{N}$. A stochastic matrix $P$ is called *irreducible* iff $\forall \ i, j = 1, \ldots, d$ non-transient states, $\exists n \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$. 
Finite State Markov Chains

Definition
A stochastic process \( \{ X_n : \Omega \rightarrow \{1, \ldots, d\} \} \) \( n \geq 0 \) is called a Markov process iff \( \exists \ P \in M_{d \times d}(\mathbb{R}) \) stochastic matrix such that \( \forall i, j = 1, \ldots, d, \forall n \in \mathbb{N}, \ P [ X_{n+1} = j | X_n = i ] = P_{i,j} \). \( P \) is called the transition probability matrix of \( \{ X_n \} \) \( n \geq 0 \).

Definition
A process \( \{ X_n \} \) \( n \geq 0 \) is called stationary iff all random variables \( X_n \) have the same distribution vector \( q \in \Delta^{d-1} = \{ (x_1, \ldots, x_d) : x_i \geq 0, \sum_{i=1}^{d} x_i = 1 \} \).
Kolmogorov Extension Theorem

Assume $P \in M_{d \times d}(\mathbb{R})$ is a stochastic matrix, $q \in \Delta^{d-1}$. Let $\sigma : \Omega(P) \rightarrow \Omega(P)$ be the shift on the Bernoulli space $\Omega(P) = \{ \omega \in \{1, \ldots, d\}^\mathbb{N} : \forall n \in \mathbb{N}, P_{\omega_n,\omega_{n+1}} > 0 \}$.

Theorem (Kolmogorov)

The function $\mu_{P,q}$ defined on the cylinders of $\Omega(P)$ by

$$\mu_{P,q} \left( \{ \omega_0 \} \times \cdots \times \{ \omega_n \} \times \{1, \ldots, d\} \times \cdots \right) = q_{\omega_0} \, P_{\omega_0,\omega_1} \cdots P_{\omega_{n-1},\omega_n}$$

extends uniquely to a measure on $\Omega(P)$, which is $\sigma$-invariant iff $q^t \, P = q$.

Let $X_0 : \Omega(P) \rightarrow \{1, \ldots, d\}$, $X_0(\omega) = \omega_0$. Then $
\{X_n = X_0 \circ \sigma^n\}_{n \geq 0}$ is a Markov process on the probability space $(\Omega(P), \mu_{P,q})$, with transition probability matrix $P$. Furthermore, $
\{X_n\}_{n \geq 0}$ is stationary iff $q^t \, P = q$. 

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Notes on Furstenberg’s paper “Noncommuting Random products"
Finite State Irreducibility

Theorem

The following statements are equivalent:

1. $\sigma : \Omega(P) \to \Omega(P)$ is ergodic w.r.t. $\mu_{P,q}$,
2. $P$ is irreducible,
3. $P_\infty = \lim_{n \to \infty} P^n$ exists, and has constant columns,
4. there is a unique $x \in \Delta^{d-1}$ such that $x^t P = x$
Absolutely Continuous Random Walk

\( \mu \in \mathcal{P}(G) \) determines the random walk
\[ \hat{\mu} : M \to \mathcal{P}(M), \quad x \mapsto \hat{\mu}_x := \mu \ast \delta_x. \]

\( \mu \ll \text{Haar measure on } G \Rightarrow \hat{\mu}_x \ll m, \text{ the Riem. meas. on } M \)

We write \( x \xrightarrow{\hat{\mu}} y \iff \exists n \in \mathbb{N} \text{ such that } \frac{d\mu^n_x}{dm}(y) > 0. \)
\( x \in M \) is called transient \( \iff x \xrightarrow{\hat{\mu}} x \) does not hold

**Definition**

The random walk \( \hat{\mu} : M \to \mathcal{P}(M) \) is said to be **irreducible** \( \iff \) there is \( C \subseteq M, \text{ measurable set, such that} \)

- \( 1. \ x \xrightarrow{\hat{\mu}} y \text{ and } x \xrightarrow{\hat{\mu}} y, \text{ for every } x, y \in C, \)
- \( 2. \ x \text{ is transient, for almost every } x \in M - C. \)
Continuous State Irreducibility

Theorem (Doeblin)

If $\hat{\mu} : M \to \mathbb{P}(M)$ is an absolutely continuous random walk then the following statements are equivalent:

1. $\hat{\mu}$ is irreducible,
2. $\hat{\mu}_\infty(A) = \lim_{n \to \infty} \hat{\mu}_x^n(A)$ exists, $\forall x \in M$, $A \subset M$,
3. there is a unique measure $\nu \in \mathbb{P}(M)$ such that $\nu = P_\mu(\nu)$
Let $\mu \in \mathcal{P}(G)$, $\mu \ll \text{Haar measure on } G$.

**Definition**

We say that $x \in M$ is a $\mu$-boundary point iff $\exists \ g \in G^\mathbb{N}$ density point of $\mu^\mathbb{N}$ such that: $(\ast) \ \delta_x = \lim_{n \to \infty} g_n \cdots g_1 g_0 \mu$.

Define $W^s(g, x) = \{ z \in M : x = \lim_{n \to \infty} g_n \cdots g_1 g_0 z \}$

$(\ast) \ \Rightarrow \ \mu(W^s(g, x)) = 1$

$\Rightarrow \ \ W^s(g, x) \subseteq \{ z \in M : z \overset{\hat{\mu}}{\sim} x \}$

$\Rightarrow \ \mu$ is irreducible
A-cocycles finite dimension

Let $M$ be a boundary of $G$.

An A-cocycle is any function $\rho : G \times M \to \mathbb{R}$ such that

1. $\rho(k, x) = 0$, for every rotation $k \in G$
2. $\rho(g g', x) = \rho(g, g' x) + \rho(g', x)$, for $g, g' \in G, x \in M$

$\mathcal{W}_G(M)$ denotes the space of all A-cocycles.

**Theorem (2)**

$\mathcal{W}_G(M)$ has finite dimension.

$\dim \mathcal{W}_G(B(G)) = \dim A$. 
An isomorphism

\[ \text{Hom}(S, \mathbb{R}) = \{ \chi : S \rightarrow \mathbb{R} : \chi \text{ is a group character} \} \]

Take \( x_0 \in B(G) \), and let
\[ H = \{ g \in G : g x_0 = x_0 \} \]
be the isotropy group at \( x_0 \).

Theorem

There is an isomorphism
\[ \Phi : \mathcal{W}_G(B(G)) \rightarrow \text{Hom}(S, \mathbb{R}) \cong \text{Hom}(A, \mathbb{R}), \quad \rho \rightarrow \chi \]
such that

\[ \chi(s) = \rho(s, x_0), \quad \forall s \in S \]
\[ \chi(\Pi_S(g k)) = \rho(g, k x_0), \quad \forall g \in G, k \in K \]

where \( \Pi_S : G = K S \rightarrow S \) is the projection \( \Pi_S(k s) = s \).
A Basis of cocycles

\[ \rho_i \leftrightarrow \chi_i, \quad \chi_i(a) = \log a_{i,i} \]

Consider the flag \( k \times_0 = (V_1, \ldots, V_i) \) where \( V_j = \langle k_1, \ldots, k_j \rangle \)

\[ a_{i,i} = \text{norm of } g : V_i/V_{i-1} \to gV_i/gV_{i-1} \]

\(~ i^{\text{th}} \text{ most largest expansion of } g~\)
A-Cocycles’ Pullbacks

Assume $M, M'$ are boundaries of $G$, and $f : M \to M'$ is a $G$-equivariant map.

**Definition**
The *pullback* of $\rho \in \mathcal{W}_G(M')$ is the function $f^* \rho : G \times M \to \mathbb{R}$ defined by $(f^* \rho)(g, x) := \rho(g, f(x))$.

**Theorem**
*The pullback is an injective linear map* $f^* : \mathcal{W}_G(M') \to \mathcal{W}_G(M)$. 
The Rank of a Boundary

Assume $M$ is a boundary of $G$.

**Definition**

We call *rank* of $M$ to the dimension of $\mathcal{W}_G(M)$.

**Theorem**

For every $1 \leq i < d$, the boundary $\mathcal{F}_{d,i}$ has rank $i$.

$$p^{d-1} = \mathcal{F}_{d,1} \leq \mathcal{F}_{d,2} \leq \cdots \leq \mathcal{F}_{d,d-1} = \mathcal{F}_{d,d} = B(G).$$
Some A-cocycles in Flag manifolds

$\rho_1 = $ logarithm-norm in $\mathbb{P}^{d-1}$

$\rho_i = $ logarithm of $i^{th}$ largest expansion in $\mathcal{F}_{d,i}$

$\rho_{\nu,i} = \rho_1 + \ldots + \rho_i$ logarithm of $i$-volume in $\mathcal{G}_{d,i} \leq \mathcal{F}_{d,i}$

$\rho_{J,i} = $ logarithm of jacobian in $\mathcal{F}_{d,i}$

These A-cocycles are related as follows

$$
\rho_{J,i} = -(d + i - 1) \rho_1 - (d + i - 3) \rho_2 - \cdots - (d + 1 - i) \rho_i
$$

$$
\rho_{J,d} = -(2d - 1) \rho_1 - (2d - 3) \rho_2 - \cdots - 3 \rho_{d-1} - \rho_d
$$

$$
\rho_{J,1} = -d \rho_1
$$
Spherical Functions

Definition
A function $\psi : G \to \mathbb{R}$ is called left uniformly continuous (l.u.c.) iff $\lim_{g' \to e} \max_{g \in G} |f(g' g) - f(g)| = 0$.

Definition
A function $\psi : G \to \mathbb{R}$ is called spherical iff $\psi$ is l.u.c. and for ever $g_1, g_2 \in G$, $\int_K \psi(g_1 k g_2) \, dk = \psi(g_1) + \psi(g_2)$.

Each spherical function $\psi : G \to \mathbb{R}$ satisfies $\psi(e) = 0$ and $\psi(k_1 g k_2) = \psi(g)$, for $k_1, k_2 \in K$, $g \in G$.

$\mathcal{V}_G$ shall denote the space of all spherical functions $\psi : G \to \mathbb{R}$. 
Assume $M$ is a boundary of $G$, and let $m$ denote the normalized $G$-invariant Riemannian measure in $M$. Define a linear map $\Psi : \mathcal{W}_G(M) \rightarrow \mathcal{V}_G$,

$$\Psi : \rho \mapsto \psi_\rho, \quad \psi_\rho(g) = \int_M \rho(g, x) \, dm(x).$$

**Theorem (3)**

Let $M = B(G)$ be the maximal boundary. Then $\Psi : \mathcal{W}_G(M) \rightarrow \mathcal{V}_G$ is a linear isomorphism.
Theorem (4)

If M is a boundary of G, for every measure \( \mu \in \mathcal{P}(G) \) of class \( B_1 \), every i.i.d. \( G \)-valued process \( \{X_n\} \) with distribution \( \mu \), every cocycle \( \rho \in \mathcal{W}_G(M) \), and every \( x \in M \), with probability 1,

\[
\lim_{n \to \infty} \frac{1}{n} \rho(X_{n-1} \cdots X_1 X_0, x) = \alpha_\mu(\rho).
\]

For \( G = SL(d, \mathbb{R}) \) theorem (4) reduces to Oseledet’s theorem. This is so for the log-norm cocycle \( \rho_1 \) on \( \mathbb{P}^{d-1} \), but also for

\[
\rho_{\mathbb{V}, i}(g, x) = \log \| (\wedge^i g) x \| \quad \text{the logarithm of } i\text{-volume cocycle, on the Grassman boundary } \mathcal{G}_{d,i} \subset \mathbb{P}(\wedge^i \mathbb{R}^d).
\]
Proof of Thm (4)

Denote by $\text{LLN}(M, \rho)$ the statement of theorem (4) for the boundary $M$ and the A-cocycle $\rho$. Then

$$\text{LLN}(M, \rho_i), \quad \forall \ i \quad \Rightarrow \quad \text{LLN}(M, \sum_i \rho_i)$$

$$f : M \to M' \text{ is } G\text{-equivariant and } \rho = f^* \rho'$$

$$\Rightarrow \quad \text{LLN}(M, \rho) \quad \Leftrightarrow \quad \text{LLN}(M', \rho')$$

The general case follows because any cocycle $\rho$ on a boundary $M$ is a linear combination of pullbacks of the volume cocycles $\rho_{V,i}$ on the Grassman boundaries $\mathcal{G}_{d,i}$, for $i = 1, \ldots, d$. 
Proofs

Formula for the Largest Lyapunov Exponent

Theorem (5)

Given $\mu \in \mathcal{P}(\text{SL}(d, \mathbb{R}))$ with compact support such that

$$\int \log \|g\| \, d\mu(g) < \infty,$$

let $G$ be the closed subgroup generated by the support of $\mu$. If $G$ is irreducible then for every, every i.i.d. $\text{SL}(d, \mathbb{R})$-valued process $\{X_n\}$ with distribution $\mu$, and every $x \in \mathbb{P}^{d-1}$, with probability 1,

$$\lim_{n \to \infty} \frac{1}{n} \log \|X_{n-1} \cdots X_1 X_0 x\| = \alpha(\mu),$$

where $\alpha(\mu) = \int_G \int_{\mathbb{P}^{d-1}} \log \|g x\| \, d\nu(x) \, d\mu(g)$ for every measure $\nu \in \mathcal{P}(\mathbb{P}^{d-1})$ such that $\mu \ast \nu = \nu$. 
Proofs

Largest Lyapunov Exponent: Proof of Thm (5)

\[ P_\mu(P^{d-1}) = \{ \pi \in P(P^{d-1}) : \mu \ast \pi = \pi \} \]

Define \( F : G^N \times P^{d-1} \rightarrow G^N \times P^{d-1} \) by \( F(g, x) = (\sigma(g), g_0x) \)

\( \pi \in P_\mu(P^{d-1}) \Rightarrow \mu \times \pi \) is \( F \)-invariant

\( \pi \) is an extremal point of \( P_\mu(P^{d-1}) \) \( \Rightarrow \mu \times \pi \) is ergodic w.r.t. \( F \)

In this case we shall say that \( \pi \) is **ergodic**.

\[ \hat{\rho} : G^N \times P^{d-1} \rightarrow \mathbb{R}, \quad \hat{\rho}(g, x) = \log \|g_0x\| \] is integrable w.r.t. \( \mu \times \pi \), for any measure \( \pi \in P_\mu(P^{d-1}) \)
If $\pi \in \mathcal{P}_\mu(\mathbb{P}^{d-1})$ is ergodic, by Birkhoff’s Theorem, for $(\mu^N \times \pi)$-almost every $(g, x) \in G^N \times \mathbb{P}^{d-1}$,

$$\lim_{n \to \infty} \frac{1}{n} \log \|X_{n-1} \cdots X_0 x\| = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \hat{\rho} \circ F^i(g, x)$$

$$= \int_{G^N} \int_{\mathbb{P}^{d-1}} \hat{\rho}(g, x) \, d\pi(x) \, d\mu^N(g)$$

$$= \int_{G} \int_{\mathbb{P}^{d-1}} \hat{\rho}(g, x) \, d\pi(x) \, d\mu(g) = \alpha_\pi$$
Proofs

Largest Lyapunov Exponent: Proof of Thm (5)

Let $G$ be irreducible, then there exist $v_1, \ldots, v_d \in \mathbb{R}^d - \{0\}$ linearly independent vectors for which the limit above holds.

For $\mu^\mathbb{N}$-almost every $g \in G^\mathbb{N}$ and every $x \in \mathbb{P}^{d-1}$,

$$\lim_{n \to \infty} \frac{1}{n} \log \|X_{n-1} \cdots X_0 x\| \leq \alpha_{\pi}.$$  

for every $x \in \mathbb{P}^{d-1}$ corresponds to a vector $\nu \in \mathbb{R}^d - \{0\}$ which can be written as a linear combination of $v_1, \ldots, v_d$.

Therefore, for all $\pi, \pi' \in \mathbb{P}_\mu(\mathbb{P}^{d-1})$ both ergodic,

$$\alpha_{\pi} \leq \alpha_{\pi'}, \forall \pi, \pi' \in \mathbb{P}_\mu(\mathbb{P}^{d-1})$$

and

$$\alpha_{\pi} = \alpha_{\pi'}, \forall \pi, \pi' \in \mathbb{P}_\mu(\mathbb{P}^{d-1})$$

Thus, we have

$$\alpha_{\pi} = \alpha_{\pi'}, \forall \pi, \pi' \in \mathbb{P}_\mu(\mathbb{P}^{d-1})$$

\qed
Positive Lyapunov Exponents

Assume \( G \subset SL(d, \mathbb{R}) \) is a non-compact closed subgroup such that every subgroup of \( G \) with finite index is irreducible.

**Theorem (6)**

*Given \( \mu \in P(SL(d, \mathbb{R})) \) with compact support such that \( \int \log \|g\| \, d\mu(g) < \infty \), if \( G \) is the closed subgroup generated by the support of \( \mu \) then \( \alpha(\mu) > 0 \).*

Follows from theorems (7) and (8) below
Zero Lyapunov Exponents

Definition
We say that a measure \( \nu \in \mathcal{P}(\mathbb{P}^{d-1}) \) is \( G \)-invariant iff \( g \nu = \nu \) for every \( g \in G \).

Theorem (7)
Assume \( \mu \in \mathcal{P}({\text{SL}}(d, \mathbb{R})) \) has compact support, \( \int \log \|g\| \, d\mu(g) < \infty \), and the closed subgroup \( G \subset {\text{SL}}(d, \mathbb{R}) \) generated by \( \text{supp}(\mu) \) is non-compact. Then \( \alpha(\mu) = 0 \implies \exists \nu \in \mathcal{P}(\mathbb{P}^{d-1}) \) \( G \)-invariant.
No $G$-invariant measures

Assume $G \subset \text{SL}(d, \mathbb{R})$ is a non-compact closed subgroup such that every subgroup of $G$ with finite index is irreducible.

**Theorem (8)**

There is no measure $\nu \in \mathcal{P}(\mathbb{P}^{d-1})$ which is $G$-invariant.

Assume $\pi \in \mathcal{P}(\mathbb{P}^{d-1})$ is $G$-invariant.

$\exists \ \{g_n\} \subset G, \ \exists \ V_1, V_2 \subset \mathbb{P}^{d-1}$ linear subvarieties s.t.$$
\forall x \notin V_1, \ g_n x \to y \in V_2.
$$

Because $G$ is non-compact $\exists \ g_n \in G$ s.t. $g_n \to \infty$.

Let $g_n = k'_n a_n k_n = \text{singular value decomp.}, \ a_n \in A, \ k_n, k'_n \in K$.

Assume $k_n \to k, \ k'_n \to k'$ and let $W_1 \subset \mathbb{P}^{d-1}$ be linear subvariety with all directions where the entries of $a_n$ stay bounded, while $W_2 \subset \mathbb{P}^{d-1}$ is the linear subvariety with all directions where the entries of $a_n$ tend to $\infty$. $W_1 \neq \emptyset$ because $\det g_n = 1$.

Set $V_1 = k^{-1} W_1$ and $V_2 = k' W_2$. 

No $G$-invariant measures: Proof of Thm (8)

Then $\text{supp}(\pi) \subseteq V_1 \cup V_2$. Take $\{W_1, \ldots, W_k\}$ to be a minimal collection of linear subvarieties of $\mathbb{P}^{d-1}$ such that

$$\text{supp}(\pi) \subseteq W_1 \cup \ldots \cup W_k.$$ 

Minimality $\Rightarrow \{W_1, \ldots, W_k\}$ is $G$-invariant.

Denote by $\mathcal{C}(\pi)$ the set of all such collections $\{W_i\}_i$.

$\{W_i\}_i \in \mathcal{C}(\pi)$ $\Rightarrow \{gW_i\}_i \in \mathcal{C}(\pi)$, $\forall g \in G$, because $\pi$ is $G$-invariant. $\{W_i\}_i, \{W'_j\}_j \in \mathcal{C}(\pi)$ $\Rightarrow \{W_i \cap W'_j\}_i,j \in \mathcal{C}(\pi)$, and this intersection is strictly smaller unless $\{W_i\}_i = \{W'_j\}_j$.

Whence, $H = \{g \in G : gW_i = W_i, \forall i = 1, \ldots, k\}$ is a reducible subgroup of $G$ with finite index, a contradiction. $\square$
Denote by $m$ the Riemannian ($K$-invariant) measure in $\mathbb{P}^{d-1}$.

$$\log \|g \cdot x\| = -\frac{1}{d} \log \det(D\varphi_g)_x = -\frac{1}{d} \log \frac{dg^{-1}m}{dm}(x).$$

**Theorem (9 Another Formula)**

*Given $\pi \in \mathcal{P}(\mathbb{P}^{d-1})$ such that $\mu \ast \pi = \pi$, $\pi \ll m$ and $m \ll \pi$,*

$$\alpha(\mu) = -\frac{1}{d} \int_G \int_{\mathbb{P}^{d-1}} \log \frac{dg^{-1}\pi}{d\pi}(\xi) \, d\pi(\xi) \, d\mu(g).$$
We have
\[
\alpha = \alpha(\mu) = \int \int \log \|g x\| \, d\pi(x) \, d\mu(g)
\]
\[
= -\frac{1}{d} \int \int \log \frac{dg^{-1}m}{dm}(x) \, d\pi(x) \, d\mu(g)
\]

Define now
\[
\beta = \beta(\mu) = -\frac{1}{d} \int \int \log \frac{dg^{-1}\pi}{d\pi}(x) \, d\pi(x) \, d\mu(g)
\]

Then, because \(\mu \ast \pi = \pi\),
Proofs

Positive Lyapunov Exponents: Proof of Thm 9

\[ d(\alpha - \beta) = \int \int \log \left( \frac{dg^{-1}m}{dm} \bigg/ \frac{dg^{-1}\pi}{d\pi} \right) \, d\pi(x) \, d\mu(g) \]

\[ = \int \int \log \left( \frac{dg^{-1}m}{dg^{-1}\pi} \bigg/ \frac{dm}{d\pi} \right) \, d\pi(x) \, d\mu(g) \]

\[ = \int \int \log \frac{dg^{-1}m}{dg^{-1}\pi} \, d\pi(x) \, d\mu(g) - \int \log \frac{dm}{d\pi} \, d\pi(x) \]

\[ = \int \int \log \frac{dm}{d\pi} (g \pi) \, d\pi(x) \, d\mu(g) - \int \log \frac{dm}{d\pi} \, d\pi(x) \]

\[ = \int \log \frac{dm}{d\pi} (y) \, d(\mu \ast \pi)(y) \, d\mu(g) - \int \log \frac{dm}{d\pi} \, d\pi(x) = 0 \]
Proofs

Positive Lyapunov Exponents: Proof of Thm (10)

Let $\mu \in \mathcal{P}(\text{SL}(d, \mathbb{R}))$ and assume the closed subgroup $G$ generated by $\text{supp}(\mu)$ is non-compact and such that every subgroup of $G$ with finite index is irreducible.

**Theorem (10)**

$\pi \in \mathcal{P}(\mathbb{P}^{d-1})$, $\mu \ast \pi = \pi$, $\pi \ll m$ and $m \ll \pi \Rightarrow \alpha(\mu) > 0$.

Assume $\alpha(\mu) = 0$. By Jensen's inequality, for $g \in G$,

$$0 = \int \log \frac{dg^{-1}\pi}{d\pi}(x) \, d\pi(x) \leq \log \int \frac{dg^{-1}\pi}{d\pi}(x) \, d\pi(\xi) = \log 1 = 0$$

$\Rightarrow \frac{dg^{-1}\pi}{d\pi}(x)$ is constant for $\pi$-a.e. $x \in \mathbb{P}^{d-1}$
Proofs

Positive Lyapunov Exponents: Proof of Thm (10)

\[ \Rightarrow \frac{dg^{-1}}{d\pi}(x) = 1 \text{ for } \pi\text{-a.e. } x \in \mathbb{P}^{d-1} \]

for \( \frac{dg^{-1}}{d\pi}(x) \) is a probability density.

\[ \Rightarrow g \pi = \pi \]

\[ \Rightarrow \text{Contradicts Thm (8)}. \]
Take $\mu \in \mathcal{P}(\text{SL}(d, \mathbb{R}))$ and assume $\alpha(\mu) = 0$.

$\mu = \lim_{n \to \infty} \mu_n$ with $\mu_n \ll \text{Haar measure on } G$.

$\pi = \lim_{n \to \infty} \pi_n$ with $\pi_n \ll m \ll \pi_n$, $\mu_n \ast \pi_n = \pi_n$.

Take for instance $\mu_n$ with supported on a compact neighbourhood of the orthogonal group $K$.

Then $\mu \ast \pi = \pi$, and

\[
\alpha(\mu) = \int_G \int_{\mathbb{P}^{d-1}} \log \|g \cdot x\| \, d\pi(x) \, d\mu(g)
\]

\[= \lim_{n \to \infty} \int_G \int_{\mathbb{P}^{d-1}} \log \|g \cdot x\| \, d\pi_n(x) \, d\mu_n(g)\]

\[0 = \lim_{n \to \infty} -\frac{1}{d} \int_G \int_{\mathbb{P}^{d-1}} \log \frac{dg^{-1} \pi_n(x)}{d\pi_n} \, d\pi_n(x) \, d\mu(g). \quad (1)\]
Positive Lyapunov Exponents: Lemma (1)

Lemma (1)

Given $\pi_1, \pi_2 \in \mathcal{P}(\mathbb{P}^{d-1})$, such that $\pi_1 \ll \pi_2$,

$$-\int_{\mathbb{P}^{d-1}} \log \frac{d\pi_1}{d\pi_2} d\pi_1 \geq 4^{-1} \|\pi_1 - \pi_2\|^2$$

$\|\pi\| = \sup_{|\phi| \leq 1} |\int \phi d\pi|$ denotes the total variation norm of a measure $\pi$. By Jensen’s Inequality

$$\int_{\mathbb{P}^{d-1}} \log \frac{d\pi_1}{d\pi_2} d\pi_1 \leq \log \int_{\mathbb{P}^{d-1}} \frac{d\pi_1}{d\pi_2} d\pi_1 = \log 1 = 0$$

$$\int_{\mathbb{P}^{d-1}} \sqrt{\frac{d\pi_1}{d\pi_2}} d\pi_1 \leq \sqrt{\int_{\mathbb{P}^{d-1}} \frac{d\pi_1}{d\pi_2} d\pi_1} = 1$$
Positive Lyapunov Exponents: Proof of Lemma (1)

\[ \| \pi_1 - \pi_2 \| = \int \left| 1 - \frac{d\pi_2}{d\pi_1} \right| \, d\pi_1 = \int \left| 1 - \sqrt{\frac{d\pi_2}{d\pi_1}} \right| \left| 1 - \sqrt{\frac{d\pi_2}{d\pi_1}} \right| \, d\pi_1 \]

\[ \leq \left\{ \int \left( 1 - \sqrt{\frac{d\pi_2}{d\pi_1}} \right)^2 \, d\pi_1 \right\} \left\{ \int \left( 1 + \sqrt{\frac{d\pi_2}{d\pi_1}} \right)^2 \, d\pi_1 \right\}^{1/2} \]

\[ = \left\{ \left( 2 - 2 \int \sqrt{\frac{d\pi_2}{d\pi_1}} \, d\pi_1 \right)^2 \left( 2 + 2 \int \sqrt{\frac{d\pi_2}{d\pi_1}} \, d\pi_1 \right)^2 \right\}^{1/2} \]

\[ = 2 \left\{ 1 - \left( \int \sqrt{\frac{d\pi_2}{d\pi_1}} \, d\pi_1 \right)^2 \right\}^{1/2} \]
Positive Lyapunov Exponents: Proof of Lemma (1)

\[ \leq 2 \left\{ 1 - \left( \exp \frac{1}{2} \int \log \frac{d\pi_2}{d\pi_1} d\pi_1 \right)^{2} \right\}^{1/2} \]

\[ \leq 2 \left\{ - \int \log \frac{d\pi_2}{d\pi_1} d\pi_1 \right\}^{1/2} \]

We have used Jensen’s inequality

\[ \int \frac{d\pi_2}{d\pi_1} d\pi_1 \geq \exp \int \log \frac{d\pi_2}{d\pi_1} d\pi_1 \]

and

\[ 1 - e^t \leq -t, \quad \forall \ t \in \mathbb{R} \]
Proofs

Positive Lyapunov Exponents: Proof of Thm (7)

Thus, by (1) and lemma (1),

$$0 = \lim_{n \to \infty} \int_{G} \| g^{-1} \pi_n - \pi_n \|^2 \, d\mu_n(g) .$$

\[\Rightarrow \forall g \text{ density point of } \mu, \exists g_n \to g \text{ in } G \text{ s.t.} \]

$$\| g_n^{-1} \pi_n - \pi_n \| = \| g_n \pi_n - \pi_n \| \to 0.$$ 

Take $\delta > 0$ and any $U$ neighbourhood of $g$. There is $\epsilon > 0$ s.t. for all sufficiently large $n$, $\mu_n(U) \geq \epsilon$ and $a_n < \delta$. Consider any $n$ large enough so that $a_n < \delta^2 \epsilon$. Then there is $g \in U$ such that $\| g^{-1} \pi_n - \pi_n \| < \delta$, for otherwise we would get

$$a_n = \int_{G} \| g^{-1} \pi_n - \pi_n \|^2 \, d\mu_n(g) \geq \delta^2 \epsilon > a_n.$$
Positive Lyapunov Exponents: Proof of Thm (7)

\[ \forall g \text{ density point of } \mu, \ g \pi_n - \pi_n \rightarrow 0 \text{ weakly.} \]

For any continuous function \( \phi \),

\[
|g \pi_n(\phi) - \pi_n(\phi)| \leq |g \pi_n(\phi) - g_n \pi_n(\phi)| + \|g_n \pi_n(\phi) - \pi_n(\phi)\| \\
\leq |\pi_n(L_g(\phi)) - \pi_n(L_{g_n}(\phi))| + \|g_n \pi_n - \pi_n\| \\
\leq \|L_g(\phi) - L_{g_n}(\phi)\|_{\infty} + \|g_n \pi_n - \pi_n\| \rightarrow 0
\]

where \( L_g(\phi)(x) = \phi(gx) \).
\( \forall g \in \text{supp}(\mu), \; g \pi = \pi. \)

First we prove this for every \( g \) density point of \( \mu \), using the weak continuity of the action \( G \times \mathcal{P}(\mathbb{P}^{d-1}) \to \mathcal{P}(\mathbb{P}^{d-1}) \), and the fact that \( g_n \pi_n - \pi_n \to 0 \) weakly. But \( \mu \)-density points are dense in \( \text{supp}(\mu) \).

\( \Rightarrow \) \( \pi \) is \( G \)-invariant.

Because the group \( G \) is generated by \( \text{supp}(\mu) \).