Quasi-Periodic Schrödinger Cocycles with Positive Lyapunov Exponent are not Open in the Smooth Topology

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Abstract

One knows that the set of quasi-periodic Schrödinger cocycles with positive Lyapunov exponent is open and dense in the analytic topology. In this paper, we construct cocycles with positive Lyapunov exponent which can be approximated by ones with zero Lyapunov exponent in the space of $C^l$ ($1 \leq l \leq \infty$) smooth quasi-periodic cocycles. As a consequence, the set of quasi-periodic Schrödinger cocycles with positive Lyapunov exponent is not $C^l$ open.

Keywords. Lyapunov exponent; Smooth quasi-periodic cocycles; Schrödinger operators.

1 Introduction and Results

Let $X$ be a $C^r$ compact manifold, $T : X \to X$ be ergodic with a normalized invariant measure $\mu$ and $A(x)$ be a $SL(2, \mathbb{R})$-valued function on $X$. The dynamical system: $(x, w) \to (T(x), A(x)w)$ in $X \times \mathbb{R}^2$ is called a $SL(2, \mathbb{R})$ cocycle (or cocycle for simplicity) over the base dynamics $(X, T)$. We will simply denoted it as $(T, A)$. If the base system is a rotation on torus, i.e., $X = T^m = \mathbb{R}^m \backslash \mathbb{Z}^m$, $T = T_\omega : x \to x + \omega$ with rational independent $\omega$, we call $(T_\omega, A)$ a quasi-periodic cocycle, which is simply denoted by $(\omega, A)$. If furthermore $A(x) = S_v(x)$ is of the form $S_v(x) = \begin{pmatrix} v(x) & -1 \\ 1 & 0 \end{pmatrix}$ with $v(x + 1) = v(x)$, we call $(\omega, S_v(x))$ a quasi-periodic Schrödinger cocycle.

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Mathematics Subject Classification (2010): Primary 37; Secondary 37D25
For any \( n \in \mathbb{N} \) and \( x \in X \), we denote

\[
A^n(x) = A(T^{n-1}x) \cdots A(Tx)A(x)
\]

and

\[
A^{-n}(x) = A^{-1}(T^{-n}x) \cdots A^{-1}(T^{-1}x) = (A^n(T^{-n}x))^{-1}.
\]

If the base dynamics \((X, T, \mu)\) is fixed, the (maximum) Lyapunov exponent of \((T, A)\) is defined as

\[
L(A) = \lim_{n \to \infty} \frac{1}{n} \int \log \|A^n(x)\| d\mu := \lim_{n \to \infty} \int L_n(A(x)) d\mu \in [0, \infty).
\]

\(L(A)\) measures the average growth rate of \(\|A^n(x)\|\).

The regularity and positivity of the Lyapunov exponent (LE) are the central subjects in dynamical systems. One is also interested in the problem whether or not cocycles with positive LE are open and dense. The problems turn out to be very subtle, which depend on the base dynamics \((X, T)\) and the smoothness of the matrix \(A\).

Firstly, classical Furstenberg theory [26] showed that for certain special linear cocycles over Bernoulli shifts, the largest LE is positive under very general conditions. Furstenberg and Kifer [27] and Hennion [29] proved the continuity of the largest LE of i.i.d random matrices under a condition of almost irreducibility. Kotani [41] showed that the LE of Schrödinger cocycles \(S_{E-v}\) is positive for almost every energy \(E\) if the potential \(v\) is non-deterministic. Viana [49] proved that for any \(s > 0\), the set of \(C^s\) linear cocycles over any hyperbolic ergodic transformation contains an open and dense subset of cocycles with nonzero LE; and the LE is continuous for \(SL(2, \mathbb{R})\)-cocycles over Markov shifts [44]. For other related results, see [7], [11] and [50].

When the base dynamics is uniquely ergodic (e.g., irrational rotation or skew shift on the torus), the positivity and continuity of the LE seem to be more sensitive to the smoothness of the matrix-valued function \(A(x)\). The LE was proved to be discontinuous at any non-uniform cocycles in the \(C^0\) topology by Furman [25] (Continuity at uniform hyperbolic cocycles and cocycles with zero LE is trivial). Motivated by Mañé [42, 43], Bochi [12] further proved a stronger result that any non-uniformly hyperbolic \(SL(2, \mathbb{R})\)-cocycle over a fixed ergodic system on a compact space can be approximated by cocycles with zero LE in the \(C^0\) topology, which shows that any non-uniform cocycle can not be an inner point of cocycles with positive LE in the \(C^0\) topology. For further related results, we refer to [9], [13], [14], [29], [36], [37], [40], [48].

On the other hand, there are tremendously many positive results in the analytic topology. Herman [30] introduced the subharmonicity method and showed that the LE of \(S_{E-2\lambda \cos x}\) is positive for \(|\lambda| > 1\) and all \(E\). Herman also proved the positivity of the LE for trigonometric polynomials if the coupling is large enough. The generalization to
arbitrary one-frequency non-constant real analytic potentials was obtained by Sorets and Spencer [47]. Same results for Diophantine multi-frequency were established by Bourgain and Goldstein [18] and Goldstein and Schlag [28]. Zhang [55] gave a different proof of the results in [47] and applied it to a certain class of analytic Szegő cocycle. For more references, one can see [17], [23], [39].

For the continuity of the LE, Large Deviation Theorems (LDT) is an important tool, which was first established by Bourgain and Goldstein in [18] for real analytic potentials with Diophantine frequencies. In [28], Goldstein and Schlag proved that, by some sharp version of LDT and generalized Avalanche Principle(AP), $L(S_{E-v})$ is Hölder continuous in $E$ if $\omega$ is a Diophantine, $v(x)$ is analytic and $L(S_{E-v}) > 0$. Jitomirskaya, Koslover and Schulteis [32] proved the continuity of the LE for a class of analytic one-frequency quasi-periodic $M(2, \mathbb{C})$-cocycles with singularities. We will briefly mention more results along this line at the end of this section. The continuity of the LE implies that the cocycles with positive LE are open in analytic topology. Together with the denseness result by Avila [1], one knows that the set of quasi-periodic cocycles with positive LE is open and dense in the analytic topology.

We have seen that the behavior of the LE in the $C^0$ topology is totally different from its behavior in the analytic topology. The smooth case is more subtle. Avila[1] proved, among other results, that the LE is positive for a dense subset of smooth quasi-periodic cocycles. Recently, with Benedicks–Carleson–Young’s method[8, 54], the authors [51] constructed quasi-periodic cocycles $(T_\omega, A)$ where $T_\omega$ is an irrational rotation $x \rightarrow x + \omega$ on $S^1$ with $\omega$ of bounded type and $A \in C^l(S^1, SL(2, \mathbb{R}))$, $0 \leq l \leq \infty$, such that the LE is not continuous at $A$ in the $C^l$ topology. Such an example in the Schrödinger class is also constructed in [51]. For $C^2$ cosine-like potentials, Anderson Localization and the positivity of LE has been established by Sinai [46] and Fröhlich-Spencer-Wittwer [24], also see Bjerklöv [10]. For the model in [46], Wang and Zhang [52] showed the continuity of the LE, which implies that non-uniform quasi-periodic cocycles can be inner points of smooth quasi-periodic cocycles with positive exponents. An interesting problem is whether or not quasi-periodic cocycles with positive exponent are open and dense in the smooth topology as in the analytic topology. As we mentioned before, the denseness follows from the result of Avila [1]. In this paper, we will prove that, different from the analytic case, the set of smooth quasi-periodic cocycles with positive exponent are not open in smooth topology.

The LE of quasi-periodic Schrödinger cocycles have attracted so much attention not only because of its importance in dynamical systems, but also due to its close relation with quasi-periodic Schrödinger operators. The latter has strong background in physics. The LE of Schrödinger cocycles coming from the eigenvalue equations of quasi-periodic Schrödinger operators encodes enormous information on the spectrum. It is known from Kotani theory that positive LE implies singular spectrum, and typically Anderson localiza-
tion, see [31, 38, 45]; while zero Lyapunov spectrum usually implies continuous, typically absolutely continuous spectrum. The positivity of the LE is also a starting point for many other problems in dynamical systems and spectral theory, such as h"older continuity of LE, continuity and topological structure of spectrum set. The recent developed methods, such as Green’s function estimates and Avalanche Principle, etc.(see [16]), depend crucially on the positivity of the LE.

Another related interesting question is the robustness of Anderson localization. i.e., wether or not the perturbations of a Schrödinger operator exhibiting Anderson localization still have Anderson localization? The answer is affirmative in the analytic category since the LE is continuous and thus the positivity of the LE is kept under perturbations.\(^3\)

We are interested in the question in smooth case, which is closely related to the problem whether or not the positivity of the LE is kept under perturbations in the smooth category, equivalently whether or not there exist smooth non-uniformly hyperbolic Schrödinger cocycles which can be accumulated by ones with zero LE in \(C^l\) topology \((l = 1, 2, \cdots, \infty)\). If it is the case, the nature of the spectrum of Schrödinger operators might exhibit dramatically changes under small perturbations of the potential in smooth topology.

The following is the main result of this paper.

**Theorem 1.** Consider quasi-periodic Schrödinger cocycles over \(\mathbb{S}^1\) with \(\omega\) being a fixed irrational number of bounded-type.\(^4\) For any \(0 \leq l \leq \infty\), there exists a Schrödinger cocycle \(S_v\) with arbitrarily large Lyapunov exponent and a sequence of Schrödinger cocycles \(S_{v_n}\) with zero Lyapunov exponent such that \(v_n(x) \to v(x)\) in the \(C^l\) topology. As a consequence, the set of quasi-periodic Schrödinger cocycles with positive Lyapunov exponent is not \(C^l\) open.

Theorem 1 can be obtained from Theorem 2 in the same way as in [51] to derive examples in Schrödinger cocycles from examples in \(SL(2, \mathbb{R})\) cocycles. Thus we only need to prove Theorem 2.

**Theorem 2.** Consider quasi-periodic \(SL(2, \mathbb{R})\) cocycles over \(\mathbb{S}^1\) with \(\omega\) being a fixed irrational number of bounded-type. For any \(0 \leq l \leq \infty\), there exists a cocycle \(D_l \in C^l(\mathbb{S}^1, SL(2, \mathbb{R}))\) with arbitrarily large Lyapunov exponent and a sequence of cocycles \(C_k \in C^l(\mathbb{S}^1, SL(2, \mathbb{R}))\) with zero Lyapunov exponent such that \(C_k \to D_l\) in the \(C^l\) topology. As a consequence, the set of \(SL(2, \mathbb{R})\)-cocycles with positive Lyapunov exponent is not \(C^l\) open.

**Remark 1.1.** Completely different from the result in Theorem 1, Bonatti, Gómez-Mont and Viana [15] proved that there exist Hölder continuous cocycles over Bernoulli shift

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\(^3\)More precisely, it is true for all almost all frequencies.

\(^4\)Bounded type means \(\frac{p_k}{q_k}\), the best approximation of \(\omega\), satisfies \(q_{k+1} \leq Mq_k\) for some \(M > 0\).
with positive LE which can be approximated by continuous cocycles with zero LE, but not by Hölder ones, which shows that the base dynamics plays an important role in the regularity problem of the LE.

**Remark 1.2.** Avila and Krikorian [6] showed that the LE is smooth in the space of smooth monotonic quasi-periodic cocycles. Our result shows that the monotonicity assumption in [6] is necessary, and behavior of the LE in smooth quasi-periodic Schrödinger cocycles homotopic to the identity are completely different from its behavior in the class of monotone cocycles.

The proof of Theorem 2 is constructive. Recall in [51], we have constructed a smooth cocycles $D_t$ with positive LE and a smooth cocycle $A_1$ in $\frac{1}{2k}$-neighborhood of $D_t$ in the $C^1$ topology for any given $k > 0$ such that the finite LE of $A_1$, defined by $L_{n_1}(A_1) = \frac{1}{n_1} \int_{[0,1]} \log \|A_1^n(x)\|dx$, is smaller that $(1 - \delta_2)L(D_t)$ for a fixed number $\delta_2 > 0$. As a consequence of subadditivity of finite LE, $L(A_1) < (1 - \delta_2)L(A)$. It follows that the LE is discontinuous at $D_t$. However, the construction in [51] did not tell us how small $L(A_1)$ can be. In this paper we will define a new $A_1$ somehow different from the one in [51] but satisfies the same property stated as above. Then we further locally modify $A_1$ such that the modified cocycle, say $A_2$, satisfies $\|A_2 - A_1\|_{C^1} < \frac{1}{4k}$ and $L_{n_2}(A_2) < (1 - \delta_2)L_{n_1}(A_1)$. It follows that $A_2$ is in the $\delta$-neighborhood of $A$ and $L(A_2) < (1 - \delta_2)^2L(A)$. Inductively, we locally modify $A_i$ such that the modified cocycle, say $A_{i+1}$, satisfies $\|A_{i+1} - A_i\|_{C^1} < \frac{1}{2k}$ and $L_{n_{i+1}}(A_{i+1}) < (1 - \delta_2)L_{n_i}(A_i)$, where $n_i \to \infty$ will be specified later. It follows that all $A_i$ are in the $\frac{1}{k}$-neighborhood of $D_t$ and $L(A_{i+1}) < (1 - \delta_2)^iL(D_t)$. It is easy to see that $A_i$ has a limit, say $C_k$, with $L(C_k) = 0$. Moreover, $\|C_k - D_t\|_{C^1} < \frac{1}{k}$. Theorem 2 is thus proved since $k$ is arbitrary.

We remark that $D_t$ and $C_k$ we constructed are of the forms $\Lambda R_{\phi(x)}$ and $\Lambda R_{\phi_k(x)}$ where $\Lambda = \text{diag}\{\lambda, -\lambda\}, \lambda \gg 1$ with $L(D_t) \sim \ln \lambda$ and $L(C_k) = 0$. Moreover, $\phi_k(x)$ is an arbitrarily small modification of $\phi(x)$ in an arbitrarily small neighborhood of two special points (called critical points). So a small change makes a big difference! For Schrödinger cocycles, we actually construct, for arbitrarily large but fixed $\lambda$, smooth $\psi(x)$ and $\tilde{\psi}(x)$ which are arbitrarily close to each other and slightly different only at the neighborhood of two critical points such that $L(S_{\lambda \psi(x)})$ is very big while $L(S_{\lambda \tilde{\psi}(x)}) = 0$. The result is surprising as we have even not seen any example of smooth Schrödinger cocycles of the form $S_{\lambda \psi(x)}$ with $\lambda \gg 1$ such that $L(S_{\lambda \tilde{\psi}(x)}) = 0$.

From our construction, one can see how and where to modify a cocycle so as to control the LE. This might be useful for other problems.

**More results on the continuity of the LE in the analytic topology.** When the base dynamics is a shift or skew-shift of a higher dimensional torus, the log-continuity of the LE was proved in [19] by Bourgain, Goldstein and Schlag. Recently, the result of [32] was gen-
eralized by Jitomirskaya and Marx [33] for all non-trivial singular analytic quasiperiodic
cocycles with one-frequency with application to the extended Harper’s model [34].

An arithmetic version of large deviations and inductive scheme were developed by
Bourgain and Jitomirskaya in [20] allowing to obtain joint continuity of the LE for SL(2, C)
cocycles, in frequency and cocycle map, at any irrational frequencies. This result has been
crucial in many further important achievements, such as the proof of the Ten Martini prob-
lem [4], Avila’s global theory of one-frequency cocycles [2, 3]. It was extended to multi-
frequency case by Bourgain [17] and to general M(2, C) case by Jitomirskaya and Marx
[34]. More recently, a completely different method without LDT or AP was developed by
Avila, Jitomirskaya and Sadel [5] and was applied to prove the continuity of the LE in
M(d, C), d ≥ 2. For further works, see [21], [22], [35], [39], [53].

2 The construction of \( D_l \)

We consider the case \( m = 1 \). We say a \( SL(2, \mathbb{R}) \)-matrix \( A \) is hyperbolic if \( \| A \| > 1 \).
A quasi-periodic cocycle \((\omega, A(x))\) of degree \( d \) is defined by a matrix function \( A(x) =
R_{\psi(x)} \cdot \Lambda(x) \cdot R_{\phi(x)} \) on \( \mathbb{R} \), with \( \Lambda(x + 1) = \Lambda(x) = diag\{\| A \|, \frac{1}{\| A \|}\} \), \( \psi(x + 1) = 2\pi d + \psi(x) \), \( \phi(x + 1) = 2\pi d + \phi(x) \) where \( R_{\theta} =\)
\( \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \). It is easy to
see that \((\phi(x) + \psi(x - \omega))\) is uniquely determined by \( A(x) \) up to \( 2\pi \mathbb{Z} \) and \( L(A) =
L(\Lambda(x) \cdot R_{\phi(x) + \psi(x - \omega)}) \) as \( A \) is conjugated to \( \Lambda(x) \cdot R_{\phi(x) + \psi(x - \omega)} \).

Let \( \Lambda = diag\{\lambda, \frac{1}{\lambda}\} \) with \( \lambda \gg 1 \). In this section, we will construct a sequence of
smooth cocycles \( B_k \) of the form \( \Lambda \cdot R_{\xi_k(x)} \), converging in \( C^l \) such that \( L(\lim B_k) > 0 \).
Moreover \( \xi_k(x) \) will be specially designed so that, in the next section, we can further
constructed cocycles \( C_k \) with zero Lyapunov exponent in any small neighborhood of \( B_k \).
When \( \lambda \) is big, we will see that the Lyapunov exponent of \( B_k \) crucially depends on the
local behavior, more precisely the degeneracy, of \( \xi_k(x) \) at the critical points \( \{ c : \xi_k(c) =
\frac{\pi}{2} \mod \pi \} \) due to the cancelation. The construction in this section is in principle along
the line of the construction in [51], the difference is in this paper, we use the decomposi-
tion of a matrix instead of the most expended and contracted direction of a matrix which
makes the proof more transparent.

Let \( \omega \) be a fixed irrational number and \( \frac{p_k}{q_k} \) be its best approximation. Throughout the
paper, we assume that \( \omega \) is of the bounded type, i.e., \( q_{k+1} \leq Mq_k; \epsilon > 0 \) is small. \( l \) is a
fixed positive integer reflecting the smoothness of cocycles. Let \( \lambda \) and \( N \) are large enough
so that
\[
10^l \sum_{k=N}^{\infty} \frac{\log q_{k+1}}{q_k} \leq \epsilon, \quad \lambda^{-1} \ll q_N^{\frac{2}{\epsilon}}.
\]
We define the decaying sequence \( \{ \lambda_k \} \) inductively by \( \log \lambda_k = \log \lambda_{k-1} - \frac{10 \log q_k}{q_{k-1}} \) where \( \lambda_N = \lambda \gg 1 \). It is easy to see that \( \lambda_k \) converges to \( \lambda_\infty \) with \( \lambda_\infty > \lambda^{1-\epsilon} \).

For \( k \geq N \), let \( C_0 = \{ 0, \frac{1}{2} \} \). \( I_{k,1} = [-\frac{1}{q_k}, \frac{1}{q_k}] \), \( I_{k,2} = [\frac{1}{2} - \frac{1}{q_k}, \frac{1}{2} + \frac{1}{q_k}] \) and \( I_k = I_{k,1} \cup I_{k,2} \). For \( C \geq 1 \), we denote by \( I_{k+C} = [-\frac{1}{Cq_k}, \frac{1}{Cq_k}] \), \( I_{k,2} = [\frac{1}{2} - \frac{1}{Cq_k}, \frac{1}{2} + \frac{1}{Cq_k}] \), and by \( I_k \) the set \( \{ I_{k+C} \} \cup I_{k,2} \). Denote Lebesgue measure of \( I_k \) by \( |I_k| \). For each \( x \in I_k \), let \( r_k^+(x) \) (respectively \( r_k^-(x) \)) be the smallest positive integer such that \( T^{r_k^+(x)}(x) \in I_k \) (respectively \( T^{-r_k^-(x)}(x) \in I_k \)). Let \( r_k^+ = \min_{x \in I_k} r_k^+(x) \) and \( r_k^- = \min \{ r_k^+, r_k^- \} \). Obviously, \( r_k \geq q_k \).

Moreover, from the symmetry between \( I_{k,1} \) and \( I_{k,2} \), we have \( r_k = r_k^+ = r_k^- \).

We define \( \xi_0 \) on \( I = I_1 \cup I_2 = \{ x : |x| \leq \frac{1}{2q_N} \} \cup \{ x : |x - \frac{1}{2}| \leq \frac{1}{2q_N} \} \) by

\[
\xi_0(x) = \begin{cases} 
\xi_{01}(x), & |x| \leq \frac{1}{2q_N}; \\
-\xi_{02}(x) \text{ (or } \xi_{02}(x)), & |x - \frac{1}{2}| \leq \frac{1}{2q_N};
\end{cases}
\]

(2.2)

where

\[
\xi_{01}(x) = \text{sgn}(x)|x|^{l+1}, \quad \xi_{02}(x) = \text{sgn}(x - \frac{1}{2})|x - \frac{1}{2}|^{l+1}.
\]

(2.3)

\( \xi(x) \) is a lift of a 1-periodic \( C^l \) function satisfying

\[
\xi(x) = \begin{cases} 
\xi_{01}(x), & |x| \leq \frac{1}{2q_N}; \\
-\xi_{02}(x) \text{ (or } \pi + \xi_{02}(x)), & |x - \frac{1}{2}| \leq \frac{1}{2q_N};
\end{cases}
\]

(2.4)

and \( |\xi(x)(\mod \pi)| > \frac{1}{2q_N} \) for any \( x(\mod 1) \notin I \). See Figures 1 and 2 for the picture of \( \xi(x) \).

![Figure 1: homotopic to identity](image1)

![Figure 2: nonhomotopic to identity](image2)

In the following, we will use \( c, C, C(l) \), etc., to denote universal positive constants independent of iterative steps. For any cocycle \( A(x), n \in \mathbb{Z}^+ \) and \( x \in I \), we decompose \( A^n(x) \) as \( R_{\psi_A,n}(x) \cdot \Lambda_A(x) \cdot R_{\psi_A,n}(x) \) when \( A^n(x) \) is hyperbolic in \( I \) and decompose \( A^n(T^{-n}x) \) as \( R_{\psi_A,-n}(x) \cdot \Lambda_A(-n) \cdot R_{\psi_A,-n}(x) \) when \( A^n(T^{-n}x) \) is hyperbolic in \( I \).

Let \( \xi_N(x) = \xi(x) \) defined above. Define \( B_N(x) = \Lambda R_{\frac{1}{2} - \xi_N(x)} \).
Proposition 2.1. There are $C^l$ functions $\xi_k(x)$ ($k = N + 1, N + 2, \cdots$) constructed inductively such that

1. $|\xi_k(x) - \xi_{k-1}(x)|_{C^l} \leq C(l) \cdot \lambda_k^{-2r_k} \cdot |I_k|^{-2}$. \hfill (2.5)

2. Let $B_k(x) = \lambda R_{k} \cdot \xi_k(x)$. For each $x \in I_k$, we have

\[ \|B_k^+(x)\| \geq \lambda_k^+ \] \hfill (2.6)

3. For $x \in I_k$, we have

\[ \begin{align*}
(1)_k & \quad \psi_{B_k, -r}^-(x) + \phi_{B_k, r}^+(x) - \frac{\pi}{2} = \xi_0(x) \quad \text{on } \frac{I_k}{10}; \\
(2)_k & \quad |\psi_{B_k, -r}^-(x) + \phi_{B_k, r}^+(x) - \frac{\pi}{2}| \geq \frac{1}{(20d_k)^{l+1}}, \quad x \in I_k \setminus \frac{I_k}{10},
\end{align*} \]

where $\xi_0(x)$ is defined in (2.2) and (2.3).

Remark 2.1. It is easy to see from (2.5) that $B_k$ converges to a limit $D_l$ in $C^l$-topology. Moreover, from (2.5) and (2.6) as well as Theorem 3 in [51], we have $L(D_l) \geq (1 - \epsilon) \ln \lambda$.

To prove Proposition 2.1, we first give the following Lemma 2.1.

Lemma 2.1. For any function $\sigma(x)$ defined on $S^1$, let $d_k(\sigma) = \min_{x \in I_k} \{|\sigma(x)|\}$. Assume that for any $x \in I_k$,

\[ \log \|A^1(x)\| \gg -\log d_{k+1}, \] \hfill (2.7)

where $d_{k+1} = d_{k+1}(\phi_{A, r_k^+}^+(x) + \psi_{A, -r_k^-}^- (x) - \frac{\pi}{2})$. Furthermore assume that, for $i \leq l$ and $m^\pm = r_k^\pm(x)$,

\[ \begin{align*}
&\left\{ \begin{array}{l}
\left| \frac{d^i}{dx} \phi_{A, m^+}^+(x) \right|, \quad \left| \frac{d^i}{dx} \psi_{A, -m^-}^- (x) \right| \leq C(i) \cdot d_{k+1}^{-i} \quad (1)_k \\
\left| \frac{d^i}{dx} A^{m}(x) \right| \cdot \|A^{m}(x)\|^{-1} \leq C(i) \cdot d_{k+1}^{-i}. \quad (2)_k
\end{array} \right.
\end{align*} \]

Then for $i \leq l$, $x \in I_{k+1}$ and $\hat{m}^\pm = r_{k+1}^\pm(x)$ it holds that

\[ \begin{align*}
&\left\{ \begin{array}{l}
\left| \frac{d^i}{dx} \phi_{A, \hat{m}^+}^+(x) \right|, \quad \left| \frac{d^i}{dx} \psi_{A, -\hat{m}^-}^- (x) \right| \leq C(i) \cdot d_{k+1}^{-i}, \quad (1)_{k+1} \\
\left| \frac{d^i}{dx} A^{\hat{m}}(x) \right| \cdot \|A^{\hat{m}}(x)\|^{-1} \leq C(i) \cdot d_{k+1}^{-i}. \quad (2)_{k+1}
\end{array} \right.
\end{align*} \]

Moreover, for any $i \geq 0$, $x \in I_{k+1}$, it holds that

\[ \begin{align*}
&\left| \frac{d^i}{dx^2} (\phi_{A, r_{k+1}^+} (x) - \phi_{A, r_k^+}^+(x)) \right| \leq C(i) \cdot \|A^2\|^{-2} \cdot d_{k+1}^{-i}, \\
&\left| \frac{d^i}{dx^2} (\psi_{A, -r_{k+1}^-} (x) - \psi_{A, -r_k^-}^- (x)) \right| \leq C(i) \cdot \|A^2\|^{-2} \cdot d_{k+1}^{-i}. \quad (2.8)
\end{align*} \]
The proof of Lemma 2.1 will be given in the Appendix.

**Proof of Proposition 2.1.** For each $k \geq N$ and $x \in I_k$, since

$$
\hat{f}_k(x) := (\psi_{B_{k-1},-r_k^-}^-(x) + \phi_{B_{k-1},r_k^+}^+(x)) - (\psi_{B_{k-1},-r_k^-}^+(x) + \phi_{B_{k-1},r_k^+}^-(x))
$$

usually does not vanish on $I_{k-1}$ and thus $\psi_{B_{k-1},-r_k^-}^+(x) + \phi_{B_{k-1},r_k^+}^-(x) - \frac{\pi}{2} \neq \xi_0(x)$ on $I_k$. To guarantee $(1)_k$ in Proposition 2.1, we modify $\xi_{k-1}(x)$ on $I_k$ as $\xi_k(x) = \xi_{k-1}(x) + f_k(x)$, where $C^l$ periodic function $f_k(x)$ is defined as follows

$$
f_k(x) = \begin{cases} 
\hat{f}_k(x), & x \in \frac{I_k}{10} \\
\hat{h}_k^x(x), & x \in I_k \setminus \frac{I_k}{10} \\
0, & x \in S^1 \setminus I_k 
\end{cases}
$$

where $\hat{h}_k^x(x)$ is a polynomial of degree $2l + 1$ restricted in each interval of $I_k \setminus \frac{I_k}{10}$ satisfying

$$
\frac{d^i h_k^x}{dx^i}(\pm \frac{1}{10q_k}) = \frac{d^j \hat{h}_k}{dx^j}(\pm \frac{1}{10q_k})
$$

$$
\frac{d^i h_k^x}{dx^i}(\pm \frac{1}{q_k}) = 0, \quad i = 1, 2, \quad 0 \leq j \leq l.
$$

From (2.8) in Lemma 2.1, we have

$$
|\psi_{B_{k-1},-r_k^-}^-(x) + \phi_{B_{k-1},r_k^+}^+(x)) - (\psi_{B_{k-1},-r_k^-}^+(x) + \phi_{B_{k-1},r_k^+}^-(x))|_{C^l} \leq C(l) \cdot \lambda_k^{-2r_k} \cdot |I_k|^{-l^2},
$$

(2.9)

where (2.7) is fulfilled by conclusion 2 and 3 of the induction assumption for the case $k - 1$.

In view of the definition of $f_k(x)$ we obtain

$$
|\hat{f}_k|_{C^l} \leq C(l) \cdot \lambda_k^{2r_k} \cdot |I_k|^{-l^2}.
$$

(2.10)

Let $B_k(x) = \Lambda \cdot R_{\frac{\pi}{2} - \xi_k(x)}$, then we have

**Lemma 2.2.** For $x \in I_k$, it holds that

$$
B_k^{r_k^+}(x) = B_{k-1}^{r_k^+}(x) \cdot R_{-f_k(x)}
$$

and

$$
B_k^{r_k^-}(x) (T^{-r_k^-}x) = B_{k-1}^{r_k^-}(x) (T^{-r_k^-}x).
$$

**Proof.** Obviously $T^i x \in S^1 \setminus I_k$ for $x \in I_k$ and $1 \leq i \leq r_k^+(x) - 1$. Since $B_k(x) = B_{k-1}(x)$ for $x \in S^1 \setminus I_k$, we have that

$$
B_k^{r_k^+}(x) = B_{k-1}^{r_k^+}(x) \cdot (B_{k-1}^{-1}(x) B_k(x)), \quad x \in I_k.
$$
From the definition, we have $B_k(x) = B_{k-1}(x) \cdot R_{\xi_{k-1}(x)}$, which implies $B_{k-1}^{-1}(x)B_k(x) = R_{\xi_{k-1}(x)} - \xi_k(x)$. Thus we obtain the first equation in Lemma 2.2. Similarly we can prove the second one.

**Lemma 2.3.** It holds that

$$f_k(x) = (\psi_{B_{k-1}, r_k}(x) + \phi_{B_{k-1}, r_k}(x)) - (\psi_{B_k, r_k}(x) + \phi_{B_k, r_k}(x)), \quad x \in I_k.$$  

**Proof.** Since a rotation does not change the norm of a vector, for a hyperbolic matrix $A$ and a rotation matrix $R_{\theta}$, we have

$$\phi_{A, R_{\theta}} = \phi_A + \theta. \quad (2.11)$$

From Lemma 2.2, we have

$$\phi_{B_{k, r_k}}(x) = \phi_{B_{k-1}, r_k}(x) - f_k(x), \quad \psi_{B_{k}, r_k}(x) = \psi_{B_{k-1}, r_k}(x).$$

Thus

$$f_k(x) = (\psi_{B_{k-1}, r_k}(x) + \phi_{B_{k-1}, r_k}(x)) - (\psi_{B_k, r_k}(x) + \phi_{B_k, r_k}(x)), \quad x \in I_k,$$

which concludes the proof. \qed

**Proof of (1) and (2)**

From the definition of $f_k(x)$, we have $f_k(x) = (\psi_{B_{k-1}, r_k}(x) + \phi_{B_{k-1}, r_k}(x)) - (\psi_{B_{k-1}, r_k}(x) + \phi_{B_{k-1}, r_k}(x))$ on $I_k$, which together with Lemma 2.3 implies that for each $x \in I_k$,

$$\psi_{B_{k-1}, r_k}(x) + \phi_{B_{k-1}, r_k}(x) = \psi_{B_{k-1}, r_k}(x) + \phi_{B_{k-1}, r_k}(x) + f_k(x) = \psi_{B_{k-1}, r_k}(x) + \phi_{B_{k-1}, r_k}(x).$$

Since $\psi_{B_{k-1}, r_k}(x) + \phi_{B_{k-1}, r_k}(x) = \xi_0(x)$ on $I_k$ by induction assumption (1),

we obtain (1) in proposition 2.1.

Obviously $\lambda_k^{q_k-1} \gg \frac{q_k}{2}$. Hence (2) in Proposition 2.1 can be obtained from the induction assumption (2) and (2.10).

**Proof of conclusion 1 of Proposition 2.1.** Conclusion 1 can be obtained from (2.9).

**Proof of conclusion 2 of Proposition 2.1.** For $x \in I_k$, let $i_1(x) < i_2(x) < \cdots < i_j(x) \leq r_k$ be the returning times of $I_{k-1}$ less than $r_k$. Since $|I_k| \leq \frac{1}{2}|I_{k-1}|$ (we can make a slight modification of the definition of $I_k$ if necessary), from the symmetry between $I_{k,1}$ and $I_{k,2}$, we have that for any $x \in I_k$, we have $T^{r_k}x \in I_{k-1}$. Then we have that $i_j(x) = r_k$. Since $T^{\iota(x)}x \notin I_k$ for $s < j(x)$, $|\theta_s - \frac{\pi}{2}| \geq \frac{1}{2q_k}$, where $\theta_s = \phi_{B_{k-1}, r_k}(x) + \phi_{B_{k-1}, r_k}(x)T^{\iota(x)}x + \psi_{B_{k, r_k}(x)} - \psi_{B_{k-1}, r_k}(x)T^{\iota(x)}x$. Together with the conclusion 3 of the induction assumption for $(k-1)$-th step we have that $|\tilde{\theta}_s - \frac{\pi}{2}| \geq \frac{1}{2q_k}$, where $\tilde{\theta}_s = \phi_{B_{k-1}, r_k}(x) + \phi_{B_{k-1}, r_k}(x)T^{\iota(x)}x + \psi_{B_{k, r_k}(x)}$. Thus from the definition of $\lambda_k$, we obtain the conclusion 2 for $k$-th step by repeated applications of Lemma A.1.
Remark 2.2. In spirit, the proof of conclusion 2 of Proposition 2.1 coincides with the one of LDT.

3 The construction of $C_k(x)$

Now we start to construct a $C_k$ in any small $C^l$-neighborhood of $B_k$ such that $L(C_k) = 0$. It is obvious that $C_k \to D_l$ in $C^l$ topology. $C_k$ will be constructed as limit of a sequence of converging cocycles, say $A_{k,i}$, in any small neighborhood of $B_k$ such that $L(A_{k,i}) \to 0$ as $i \to \infty$. By the construction, we can show that $L(C_k) = \lim_{i \to \infty} L(A_{k,i}) = 0$, see Corollary 3.1. In the following, we shall simply denote $A_{k,i}$ by $A_i$.

The following lemma is of key importance for the construction:

Iterative Lemma: Let $A_0(x) = \Lambda \cdot R_{\frac{\pi}{2} - \theta_0(x)}$ satisfy that $\|A_0^{r_{n_0}(x)}(x)\| \geq \mu^{r_{n_0}(x)}$ with $\lambda \geq \mu \gg 1$, $n_0 \geq N$ and $\psi_{A_0,-r_{n_0}}(x) + \phi_{A_0,r_{n_0}}(x) - \frac{\pi}{2} = \xi_0(x)$, $x \in I_{n_0}$. Then we can find two small positive numbers $\delta_1 > \delta_2$ such that for any $i \geq 0$, there exist $A_i(x) = \Lambda \cdot R_{\frac{\pi}{2} - \theta_i(x)}$ and $n_i$, such that the following hold

\[
(P_1): \begin{cases} 
(1). \|A_i^{r_{n_i}(x)}(x)\| \geq \mu^{(1-\delta_1)^i \cdot r_{n_i}(x)} \text{ on } I_{n_i} \text{ and } \mu^{(1-\delta_1)^i \cdot q_{n_i}} \gg \frac{1}{|I_{n_i}|}; \\
(2). \|A_i^{r_{n_i}(x)}(x)\| \leq \lambda^{(1-\delta_2)^i \cdot r_{n_i}(x)} \text{ for } x \in I_{n_j} \text{ and } j \leq i; \\
(3). \bar{\mu}_{n_i} \leq \mu^{2}; \\
(4). \psi_{A_i,-r_{n_i}}(x) + \phi_{A_i,r_{n_i}}(x) - \frac{\pi}{2} = \xi_0(x) \text{ on } I_{n_i}; \\
(5). |\theta_{i+1} - \theta_i| \leq q_{n_i}^{2} \cdot \mu^{-\frac{1}{2} (1-\delta_1)^i \cdot q_{n_i}} + q_{n_i}^{-2}. \end{cases}
\]

In the above, $\bar{\mu}_{n_i} = \max_{x \in I_{n_i}} \|A_i^{r_{n_i}(x)}(x)\|^{\frac{1}{r_{n_i}(x)}}$ and $\underline{\mu}_{n_i} = \min_{x \in I_{n_i}} \|A_i^{r_{n_i}(x)}(x)\|^{\frac{1}{r_{n_i}(x)}}$.

Therefore, $\underline{\mu}_{n_i} \geq \mu^{(1-\delta_1)^i}$ and $\bar{\mu}_{n_i} \leq \lambda^{(1-\delta_2)^i}$.

The main result Theorem 2 is an easy consequence of the following corollary.

Corollary 3.1. There exists a $SL(2, \mathbb{R})$-sequence $\{C_k\}_{k=N}^{\infty}$ with $L(C_k) = 0$ such that $C_k$ tend to $D_l$ in the $C^l$ topology.

Proof. For any $k \in \mathbb{N}$, we apply Iterative Lemma by setting $A_0 = B_k$, $n_0 = q_k$ and $\mu = \lambda^{1-i}$ where $B_k$ is defined in Proposition 2.1. Hence for each $i$ we obtain $A_i$ such that $(P_1)$ holds true. By (5) and the inequality $\mu^{(1-\delta_1)^i \cdot q_{n_i}} \gg \frac{1}{|I_{n_i}|}$ in (1) of $(P_1)$, $A_i$ has a limit, say $C_k$, in the $C^l$ topology. From (2) of $(P_1)$, we obtain $\|C_k^{r_{n_j}(x)}(x)\|^{\frac{1}{r_{n_j}(x)}} \leq \lambda^{(1-\delta_2)^j}$ for any $j \leq i$ and $x \in I_{n_j}$, which by the subadditivity of Lyapunov exponent, implies $L(C_k) \leq (1-\delta_2)^j \log \lambda$ for any $j$. Let $j \to \infty$, we obtain $L(C_k) = 0$. Moreover, from (5)
of \((P_i)\) it holds that
\[
\|C_k - D_i\|_{c^l} \leq \|C_k - B_k\|_{c^l} + \|B_k - D_i\|_{c^l}
\]
\[
= \|C_k - A_0\|_{c^l} + \|B_k - D_i\|_{c^l}
\]
\[
\leq 2\left(4q_i^{1.2l} \cdot \lambda^{-((1-\epsilon)(1-\delta_i) q_i + q_i^{-2})} + \|B_k - D_i\|_{c^l}\right).
\]

In the last inequality, we use (5) and the inequality \(\mu^{(1-\delta_i)q_i} \gg \frac{1}{|\text{Int}^*|}\) in (1) of \((P_i)\). It implies \(C_k \to D_i\) in the \(c^l\) topology as \(k \to \infty\). Moreover, \(L(D_i) \geq (1 - \epsilon) \ln \lambda\) by Remark 2.1. □

**Proof of Iterative Lemma.** Let \(0 < \delta_0 \ll \max\{\frac{1}{q_l}, M^{-k_1}\}\) be a fixed number and \(\delta_1 = 8\delta_0\), \(\delta_2 = M^{-k_1} \cdot \delta_0\), where \(k_1\) is defined in Proposition 3.1.

When \(i = 1\), \((P_i)\) obviously holds true for \(A_1\) with \(\lambda \gg 1\). Assuming that \(A_1, \cdots, A_{i-1}\) have been constructed with \((P_1), \cdots, (P_{i-1})\), we will construct \(A_i\) such that \((P_i)\) holds. From (3) of \((P_{i-1})\), we have \(\|A_i^{n_i-1}(x)\| \leq \|A_i^{n_i-1}(y)\|^2\) for \(x, y \in I_{n_i-1}\).

**Step 1. Definition of \(n_i\) and \(I_{n_i}\).** Choose \(n_i \gg n_{i-1}\) such that \(\mu^{(1-\delta_i)q_i} \gg q_i^2 \gg \lambda^{2\delta_0} r_{n_i-1}\) and \(I_{n_i}\) is defined as before. The Diophantine condition implies that \(r_{n_i} \geq q_{n_i}\).

**Step 2. Modification of \(A_{i-1}\).** For our purpose, we first make a local modification for \(A_{i-1}\) on \(I_{n_i-1}\) such that there is a low platform in the image of \(\phi_{A_{i-1}, r_{n_i-1}}(x) + \psi_{A_{i-1}, -r_{n_i-1}}(x) - \frac{\pi}{2}\) for the new cocycle \(\tilde{A}_{i-1}\), see Figure 3.

Consider the sub-interval \([0, \frac{1}{q_{n_i-1}}]\) of \(I_{n_i-1}\). Define \(0 < c < \tilde{c} < d < \frac{1}{2q_{n_i-1}}\) such that \(|c| = \mu^{-2\delta_0} r_{n_i-1}, |\tilde{c}| = M^2 \cdot |d| = \frac{1}{2q_{n_i-1}}\), see Figure 3. From the definition of \(n_i\), we have \(|I_{n_i}| < 2l\).

Define
\[
e_i^0(x) = \begin{cases} 
2|c|^{l+1} - (\phi_{A_{i-1}, r_{n_i-1}}(x) + \psi_{A_{i-1}, -r_{n_i-1}}(x) - \frac{\pi}{2}), & x \in [\tilde{c}, d]; \\
0, & x \notin [c, \frac{1}{2q_{n_i-1}}] \\
\tilde{h}_i(x), & x \in [c, \tilde{c}] \cup [d, \frac{1}{2q_{n_i-1}}],
\end{cases}
\]
where \(\tilde{h}_i(x)\) are polynomials of degree \(2l + 1\) restricted on each interval and for \(0 \leq j \leq l\) satisfies
\[
\frac{d^j\tilde{h}_i}{dx^j}\left(\frac{1}{2q_{n_i-1}}\right) = 0, \quad \frac{d^j\tilde{h}_i}{dx^j}(d) = \frac{d^j(2|c|^{l+1} - (\phi_{A_{i-1}, r_{n_i-1}} + \psi_{A_{i-1}, -r_{n_i-1}} - \frac{\pi}{2}))}{dx^j}(d),
\]
\[
\frac{d^j\tilde{h}_i}{dx^j}(c) = 0, \quad \frac{d^j\tilde{h}_i}{dx^j}(\tilde{c}) = \frac{d^j(2|c|^{l+1} - (\phi_{A_{i-1}, r_{n_i-1}} + \psi_{A_{i-1}, -r_{n_i-1}} - \frac{\pi}{2}))}{dx^j}(\tilde{c}).
\]

\(e_i^1(x)\) on the subinterval \([\frac{1}{2q_{n_i-1}}, \frac{1}{2q_{n_i-1}}]\) of \(I_{n_i-1}\) is defined similarly. Let \(e_i(x) = e_i^0(x) + e_i^1(x)\). We have the following estimates for \(e_i(x)\).
Lemma 3.1. It holds that $|e_i(x)|c^l \leq C \cdot q_{n_{i-1}}^{-2}$.

Proof. From (1) in $(P_{i-1})$ we have $\mu_{n_{i-1}} \geq \mu^{(1-\delta_1)^{i-1} \cdot q_{n_{i-1}}} \geq \frac{1}{|I_{n_{i-1}}|}$. Then From (4) in $(P_{i-1})$ and the definition of $c$, it holds for $0 \leq j \leq l$ that

$$|(2c)^{l+1} - (\phi_{A_{i-1},r_{n_{i-1}}} + \psi_{A_{i-1},-r_{n_{i-1}}} - \frac{\pi}{2})| \leq C \cdot q_{n_{i-1}}^{-2(l+1-j)}.$$ 

Hence from Cramer’s rule we have that $|\tilde{h}_i(x)|c^l \leq C \cdot q_{n_{i-1}}^{-2}$. Consequently, $|e_i(x)|c^l \leq C \cdot q_{n_{i-1}}^{-2}$.

Let $\theta_i = \theta_{i-1} + e_i(x)$ and $\bar{A}_{i-1} = \Lambda \cdot R \frac{d_{i-1}}{d_{i-1}}$, we have $\psi_{\bar{A}_{i-1},-r_{n_{i-1}}} + \phi_{\bar{A}_{i-1},r_{n_{i-1}}} - \frac{\pi}{2}$ on the part $[0, \frac{1}{q_{n_{i-1}}}]$ of $I_{n_{i-1}}$ is of the shape in Figure 3.

Step 3. The estimate on the lower bound.

**Lemma 3.2.** Let $A_{i,0} = \Lambda \cdot R \bar{d}_i := \Lambda \cdot R \bar{d}_{i,0}$ satisfy $\|A_{i,0}^{r_{n_{i-1}}-1}(x)\| \geq \nu_0^{r_{n_{i-1}}-1}(x)$ for $x \in I_{n_i}$ with $\nu_0 = \mu^{(1-\delta_1)^{i-1}}$. Then for any $n_i - n_{i-1} \geq j \geq 1$, there exist $\theta_{i,j}$ and $A_{i,j} = \Lambda \cdot R \frac{d_{i,j}}{d_{i,j}}$ such that the following properties hold true:

$$(\bar{P}_{i,j}) : \begin{cases} 
(1) & \|A_{i,j}^{r_{n_{i-1}}+j}(x)\| \geq \nu_j^{r_{n_{i-1}}+j}(x) \text{ on } I_{n_j}; \\
(2) & \phi_{A_{i,j},r_{n_{i-1}}+j}(x) + \psi_{A_{i,j},-r_{n_{i-1}}+j}(x) - \frac{\pi}{2} = \theta_{i,0}(x) \text{ on } I_{n_j}; \\
(3) & |	heta_{i,j} - \theta_{i,j-1}|c^l \leq r_{i,j} \cdot \nu_j^{q_{n_{i-1}}+j-1}, \quad r_{i,j} \approx \max\{\nu_{j-1}^{2\delta_d q_{n_{i-1}}}, q_{n_{i-1}+j}^{2\delta_d^2}\},
\end{cases}$$

where $\nu_j$ are iteratively defined by

$$\nu_j = \nu_0 \cdot \nu_0^{\delta_d (\sqrt{2}^{(j-1)} + \cdots + \sqrt{2}^{-1})2(l+1)} \geq \nu_0^{(1-5\delta_d(1+1))} = \mu^{(1-\delta_1)^{j}}.$$
Let \( \mu_{i,j} = \min_{x \in I_{i,j}} \| (A_{i,j}^{-n_{i-1}+j}(x)) \|^{r_{n_{i-1}+j}} \) for any \( j \leq n_i - n_{i-1} \). We have \( \mu_{i,j} \geq \nu_j \) and 
\[ \mu_{n_i} = \mu_{i,n_i-n_{i-1} \geq \nu_{n_i-n_{i-1}}}. \]

**Proof.** For \( j = 1 \), from (1) of \((P_{i-1})\) and the definitions of \( \tilde{\theta}_i \) and \( \mu_0 \), we have

\[ \frac{1}{r_{n_i-1+j}(x)} \log \| A_{i,j}^{-n_{i-1}+j}(x) \| \geq \log \mu_{i,n_{i-1}} + \frac{1}{q_{n_i-1+j}} \log \mu_{i,n_{i-1}} \]

\[ = (1 - 2(l+1)\delta_0) \log \mu_{i,n_{i-1}} \geq \nu_1. \]

Thus we obtain (1). Moreover (2) and (3) can be proved by Proposition 2.1 and Lemma 2.1 with \( d_{n_i-1+j} \geq \frac{1}{q_{n_i-1+j}} \).

Assume \((\tilde{P}_{i,j})\) hold true. We will prove \((\tilde{P}_{i,j+1})\). Define \( \tilde{\theta}_{i,j+1}(x) \) by modifying \( \theta_{i,j}(x) \) in the same way as we define \( \xi_{k+1} \) by modifying \( \xi_k \) in Proposition 2.1. Applying Lemma 2.1 with \( d_{n_i-1+j} \geq \min \{ -2(l+1)\delta_0, q_{n_i-1+j} \} \), we get (2) and (3).

Now we prove (1). In case that \( |\tilde{c}| < |I_{n_i-1+j}| \), we have \( q_{m+1} \geq \sqrt{2} \cdot q_m \) for each \( m \), and thus

\[ \frac{1}{r_{n_i-1+j}(x)} \log \| A_{i,j+1}^{-n_{i-1}+j}(x) \| \geq \log \nu_j + \frac{1}{q_{n_i-1+j+1}} \cdot \log \mu_{i,n_{i-1}} \]

\[ \geq (1 - \delta_0(\sqrt{2}^{j-1} + \cdots + \sqrt{2}^{-1} + 1) \cdot 2(l+1)) \log \nu_0 - \delta_0 \cdot \sqrt{2}^{j-1} \cdot 2(l+1) \log \nu_0 \]

\[ = \log \nu_{j+1}. \]

Now we consider the case \( |\tilde{c}| \geq |I_{n_i-1+j}| \). Let \( j^* \) be the smallest integer such that \( |I_{q_{n_i-1+j^*}}| \leq |\tilde{c}| \). (Obviously, \( j^* \) depends on \( n_i \) and we can choose \( n_i \) large enough such that \( j^* \ll n_i \).)

Since for any \( s \), it holds that \( M^2 \cdot |I_{s+1}| \geq |I_s| \). Thus from the definition of \( |c| \) and \( |\tilde{c}| \), we have \( |I_{q_{n_i-1+j^*}}| \geq |c| \) since \( |I_{q_{n_i-1+j^*+1}}| \geq |\tilde{c}| \). Notice that \( \nu_{j_{n_i-1+j^*}} \gg q_{n_i-1+j^*+1} \). We construct \( \psi_{i,j^*+m} \) and \( A_{i,j^*+m} = \Lambda \cdot R_{\psi_{i,j^*+m}} \) as in Proposition 2.1, such that

\[ \mu_{i,n_i-1+j^*+m} \geq \nu_{j^*+m}, \quad \text{for} \quad m \geq 1, \]

which thus implies (1). \( \square \)

Define \( \tilde{\theta}_i(x) = \theta_{i,j^*+m^*}(x) \) and \( A_i(x) = \Lambda \cdot R_{\frac{x}{2} - \tilde{\theta}_i(x)} \), where \( m^* = n_i - n_{i-1} - j^* \). Then (1) of \((P_i)\) can be proved by (1) in \((\tilde{P}_{i,j})\). From the inequality \( 0 < \delta_0 \ll \frac{1}{P_r} \), (5) of \((P_i)\) can be proved by (3) in \((\tilde{P}_{i,j})\) and Lemma 3.1. (4) of \((P_i)\) is obvious from the construction of \( A_{i,j} \).

**Step 4. The estimate on the upper bound.**

Now we prove (2) of \((P_i)\), i.e., an upper bound estimate for the Lyapunov exponent. We need the following proposition in [51]:
Proposition 3.1. Let $I_1$ be a small interval in $S^1$, $I_2 = I_1 + 1/2$, $I = I_1 \cup I_2$. Let

$$
\min r(x) = \min_{x \in I} \min \{i > 0 | T^i x \mod 2\pi \in I\},
$$

$$
\max r(x) = \max_{x \in \frac{1}{10} I_1} \min \{i > 0 | T^i x \mod 2\pi \in \frac{1}{10} I_1\}.
$$

Then there exists $k_1 \in \mathbb{N}$ such that $M^{-k_1} \leq \frac{\min r(x)}{\max r(x)} \leq 1$.

Obviously, we have

$$
|\phi_{A_i, r_{n_{i-1}}}(x) + \psi_{A_i, r_{n_{i-1}}}(x) - \frac{\pi}{2}| \leq 2|c|^\frac{1}{2} = 2\mu_{n_{i-1}}^{-2}q_{n_{i-1}}
$$

for $x \in [0, \frac{1}{2q_{n_{i-1}}}]$. Apply Proposition 3.1 with $I_1 = [0, \frac{1}{2q_{n_{i-1}}}]$. Then from (3.12) and Lemma A.1 we have that

$$
\bar{\mu}_{n_i} \leq \bar{\mu}_{n_{i-1}} \cdot \mu_{n_{i-1}}^{-2}M^{-k_1}.\delta_0.
$$

Subsequently (3) of $(P_{i-1})$ and the definition of $\delta_2$ imply that

$$
\bar{\mu}_{n_i} \leq \bar{\mu}_{n_{i-1}}^{1-\delta_2} \leq \lambda^{(1-\delta_2)^i}.
$$

Step 5. The comparison between the lower and upper bounds. For (3) of $(P_0)$, the upper bound for $w_0$ can be achieved by choosing $\lambda \gg N \gg 1$. For $i > 0$, we have the following argument. For any $x, y$ with $|x - y| \leq \frac{1}{q_{n_i}}$ we have that $\|A_i(x) - A_i(y)\|_{C^r} \leq \frac{C}{q_{n_i}}$. From (1) of $(P_{i-1})$ it holds that $\mu^{(1-\delta_1)^i}q_{n_{i-1}} \gg \frac{1}{|I_{n_{i-1}}|}$. Once $n_{i-1}$ is determined, for any $\bar{\varepsilon} > 0$, we can find $n_i \gg \bar{n}_i \gg n_{i-1}$ such that $|\bar{I}_{n_i}| \ll |c|$ and for any $x, y$ with $|x - y| \leq \frac{1}{q_{n_i}}$, it holds that

$$
(1 - \bar{\varepsilon}) \cdot L_{r_{\bar{n}_i}}(A_i(y)) \leq L_{r_{\bar{n}_i}}(A_i(x)) \leq (1 + \bar{\varepsilon}) \cdot L_{r_{\bar{n}_i}}(A_i(y)),
$$

$$
\|\phi_{A_i, r_{\bar{n}_i}}(x) - \phi_{A_i, r_{\bar{n}_i}}(y)\|_{C^r} + \|\psi_{A_i, r_{\bar{n}_i}}(x) - \psi_{A_i, r_{\bar{n}_i}}(y)\|_{C^r} \leq \bar{\varepsilon}.
$$

Apply the inductive process for $A_i$ from step $\bar{n}_i$ to $n_i$ for $x \in I_{n_i}$. Thus similar to Remark 2.1, if $\bar{n}_i$ is large enough such that $\mu^{(1-\delta_1)^i}q_{n_i} \gg \frac{1}{|I_{n_i}|}$, we have that for any $x \in I_{n_i},$

$$
\prod_j \|A_i^{r_{n_{i,j}}}(x_j)\| \geq \|A_i^{r_{n_{i,j}}}(x)\| \geq \left(\prod_j \|A_i^{r_{n_{i,j}}}(x_j)\|\right)^{1-\bar{\varepsilon}},
$$

where $x_j$’s are the points on orbits of $x$ when returning to $I_{\bar{n}_i}$ and $\bar{n}_{i,j}$’s are the corresponding returning times. Hence for any $x, y \in I_{n_i}$ we have

$$
(1 - 2\bar{\varepsilon})L_{r_{n_i}}(A_i(x)) \leq L_{r_{n_i}}(A_i(y)) \leq (1 + 2\bar{\varepsilon})L_{r_{n_i}}(A_i(x)).
$$

Then it follows that (3) of $(P_i)$ holds true. This ends the proof of Iterative Lemma. \qed
4 The proof for the $C^\infty$ case

In this section, we will prove Theorem 1 and 2 for the $C^\infty$ case. The basic idea is same as the one for the finitely smooth case. Essentially, we only need to modify cocycles in $C^\infty$ category. We will focus on the difference between the two cases. First we follow the steps in section 3 to construct a sequence of $C^\infty$ cocycle which is $C^1$-convergent. Then we will prove that it actually converges in $C^\infty$ topology.

Assume $\lambda \gg e^{q_n^a+1} \gg 1$ with $0 < a < \frac{1}{10}$. For $n > N$, let $\lambda_n^{q_n+1} = \lambda_n^{q_n+1} \cdot e^{-(10q_n^a+a)}$ with $\lambda_N = \lambda$. From the definition of $\lambda$, we have $\frac{\lambda_n^a}{\lambda} \geq \frac{\lambda_{n-1}^a}{\lambda} \cdot e^{-(10q_n^a+a)} \geq \lambda_{n-1}^{q_n^a} \cdot e^{-(10q_n^a+a)} \geq \cdots \geq \lambda_n^{q_n^a} \cdot e^{-(10q_n^a+a)} \geq \lambda^{(1-\varepsilon/2)q_n}$ for some small positive $\varepsilon$ if $\lambda \gg 1$ and $N \gg 1$. It implies that $\lambda_n$ decrease to $\lambda_\infty > \lambda^{1-\varepsilon}$.

Construction of $B_N(x)$ Let

(a)

$$
\xi_0(x) = \begin{cases} 
\xi_{01}(x) & \text{for } |x| \leq \delta, \\
\xi_{02}(x) (\text{or } -\xi_{02}(x)) & \text{for } |x - 1/2| \leq \delta,
\end{cases}
$$

where $\xi_{01}(x) = \text{sgn}(x) e^{-\frac{1}{|x|^2}}$ and $\xi_{02}(x) = \text{sgn}(x - 1/2) e^{-\frac{1}{|x-1/2|^2}}$, $\delta > 0$ is a small number. Let $\xi(x)$ be a lift of a $C^\infty$ 1-periodic function satisfying

$$
\xi(x) = \begin{cases} 
\xi_{01}(x), & |x| \leq \delta; \\
-\xi_{02}(x) (\text{or } \pi + \xi_{02}(x)), & |x - 1/2| \leq \delta.
\end{cases}
$$

(4.13)

(b) $\forall |x (\text{mod } 1)| > \delta$ and $|(x - 1/2)(\text{mod } 1)| > \delta$, $|\xi(x) (\text{mod } \pi)| > e^{-\frac{1}{10}}$.

Define $\xi_N(x) = \xi(x)$ and $B_N(x) = \Lambda \cdot R_{\frac{\pi}{2} - \xi_N(x)}$.

We restate Lemma 5.1 in [51] as follows:

Lemma 4.1. For each $n \geq N$, there exist a $g_n(x) \in C^\infty$ be a 1-periodic function such that

$$
g_n(x) : \begin{cases} 
= 1, & x \in \left[0, \frac{L_n}{10}\right], \\
\in \left[0, 1\right], & x \in I_n \setminus \left[0, \frac{L_n}{10}\right] \\
= 0, & x \in S^1 \setminus I_n
\end{cases}
$$

and

$$
\left| \frac{d^r g_n(x)}{dx^r} \right| \leq q_n^{3r}, \quad 0 \leq r \leq \left[\frac{3}{q_n^a}\right].
$$

(4.14)

Using the same argument as that in finite smooth case, we have that for any $x \in I_N$,

$$
\|B_N^{\tau_N}(x)\| \geq \lambda_N^{\tau_N}(x)
$$

and

$$
|\phi_{B_N,r_N}(x) + \psi_{B_N,-r_N}(x) - \frac{\pi}{2} - \xi_0(x)| \leq \lambda_N^{-1}
$$

(4.15)

for $x \in I_N$. 

Define a 1-periodic function \( e_N(x) \in \mathcal{C}^\infty \) such that \( e_N(x) = -(\phi_{B_N,r_N}(x)+\psi_{B_N,-r_N}(x)-\frac{\pi}{2} - \xi_0(x)) \) for \( x \in I_N \).

Let \( \hat{e}_N(x) = e_N(x) \cdot g_N(x) \) and \( \xi_{N+1}(x) = \xi_N(x) + \hat{e}_N(x) \) for \( x \in \mathbb{S}^1 \). Define \( B_{N+1}(x) = \Lambda \cdot R_{\frac{\pi}{2} - \xi_{N+1}(x)} \). Obviously, \( B_{N+1}(x) = B_N(x) \cdot R_{-\hat{e}_N(x)} \). Then for any \( x \in I_N \),

\[
\|B_{N+1}^{r_N}(x)\| \geq \lambda_{N}^r(x) \quad \text{and} \quad \phi_{B_{N+1},r_N}(x) + \psi_{B_{N+1},-r_N}(x) = \phi_{B_N,r_N}(x) + \psi_{B_N,-r_N}(x) - \hat{e}_N(x) \text{, which implies} \quad \phi_{B_{N+1},r_N}(x) + \psi_{B_{N+1},-r_N}(x) - \frac{\pi}{2} = \xi_0(x) \text{ on } \frac{I_N}{10} \quad (4.15) \implies \text{that} \quad |\hat{e}_N(x)|_{C^1} \leq \lambda_{N}^{-1} \text{ in } I_N \text{. Thus we have} \quad |\phi_{B_{N+1},r_N}(x) + \psi_{B_{N+1},-r_N}(x) - \frac{\pi}{2}| \geq \frac{1}{2} \cdot e^{-(10q_N^2)^n} \quad \text{on } I_N \setminus \frac{I_N}{10} .
\]

For any \( n \geq N \), define a 1-periodic function \( e_n(x) \in \mathcal{C}^\infty \) such that

\[
e_n(x) = (\phi_{B_n,r_n}(x) + \psi_{B_n,-r_n}(x)) - (\phi_{B_n,r_{n+1}}(x) + \psi_{B_n,-r_{n+1}}(x)) \quad x \in I_n.
\]

Define \( \hat{e}_n(x) = e_n(x) \cdot g_n(x) \), \( \xi_n(x) = \xi_{n-1}(x) + \hat{e}_n(x) \) and \( B_n(x) = \Lambda \cdot R_{\frac{\pi}{2} - \xi_n(x)} \). Obviously, \( B_n(x) = B_{n-1}(x) \cdot R_{-\hat{e}_n(x)} \). Then we obtain (2.5), (2.6) of Proposition 2.1 and

\[
|\phi_{B_n,r_n}(x) + \psi_{B_n,-r_n}(x) - \frac{\pi}{2}| = e^{-|x|^{a}} \quad \text{or} \quad e^{-|x-1/2|^{a}}, \quad x \in \frac{I_n}{10}, i = 1, 2
\]

\[
|\phi_{B_n,r_n}(x) + \psi_{B_n,-r_n}(x) - \frac{\pi}{2}| \geq \frac{1}{2} \cdot e^{-(10q_n^2)^n}, \quad x \in I_n \setminus \frac{I_n}{10} .
\]

From (2.5), one easily sees that \( B_N(x), B_{N+1}(x), \ldots \), is \( C^1 \)-convergent to some \( D_\infty(x) \). Furthermore, from (2.6), the Lyapunov exponent of \( D_\infty(x) \) has a lower bound \( \log \lambda_\infty > (1 - \epsilon) \log \lambda \).

In the following, we will prove that \( B_N(x), B_{N+1}(x), \ldots \), is also convergent to \( D_\infty(x) \) in \( \mathcal{C}^\infty \)-topology.

**Lemma 4.2.** \( B_N(x), B_{N+1}(x), \ldots \), is also convergent to \( D_\infty(x) \) in \( \mathcal{C}^\infty \)-topology.

**Proof.** It is equivalent to prove that \( \xi_n(x), n = N, N+1, \ldots \) is \( \mathcal{C}^\infty \)-convergent. From the definition of \( \xi_n(x) \), we have \( \xi_n(x) - \xi_{n-1}(x) = \hat{e}_n(x) \). From the definition of \( \hat{e}_n(x) \), it is sufficient to estimate \( e_n(x) \) and \( g_n(x) \). Since \( e_n(x) \) is determined by \( \phi_{B_n,r_n}(x) - \phi_{B_n,r_{n+1}}(x) \) and \( \psi_{B_n,-r_n}(x) - \psi_{B_n,-r_{n+1}}(x) \), with the help of Lemma 2.1, we have

\[
\left| \frac{d^r e_n(x)}{dx^r} \right| \leq C(r) \cdot \lambda_n^{-q_n^{a-1}}, \quad 0 \leq r \leq [\frac{3}{2}] .
\]

Note that \( C(r) \) is independent of \( n \). Thus for any fixed \( R \in \mathbb{N} \), we can choose \( n \) large enough such that \( C(r) \leq \lambda_n^{\frac{1}{2} q_n^{a-1}} \) for any \( r \leq R \). This together with (4.14) ends the proof. \( \square \)

**Construction of \( C_k(x) \)** Next we will construct the sequence \( C_k(x)(k = N, N+1, \ldots) \) with \( L(C_k) = 0 \) such that \( \mathcal{C}^\infty \) converge to \( D_\infty \).
Consider the sub-interval \([0, \frac{1}{q^4 n_i-1}]\) of \(I_{n_i-1}\). Define \(0 < c < \bar{c} < d < \frac{1}{q^4 n_i-1}\) such that 
\[
|c| = (2\delta_0 \cdot r_{n_i-1} \cdot \log \mu_{n_i-1})^{-1/a},
\]
\[
|\bar{c}| = M^2 \cdot |c|, d = \frac{1}{2q^4 n_i-1}.
\]
Let \(n_i\) be sufficiently large such that \(I_{n_i} \not\subseteq [0, c]\).

Define
\[
\bar{e}_i(x) = \begin{cases} 
  e^{-|x|^a} - (\phi_{A_{n_i-1}, r_{n_i-1}}(x) + \psi_{A_{n_i-1}, -r_{n_i-1}}(x)), & x \in [\bar{c}, d], \\
  0, & x \notin \left[\bar{c}, \frac{1}{q^4 n_i-1}\right], \\
  \bar{h}_i^\pm(x), & x \in [\bar{c}, d] \cup [d, \frac{1}{q^4 n_i-1}],
\end{cases}
\]
where \(\bar{h}_i^\pm(x)\) is of a \(C^\infty\) connection between the parts in \([0, c]\) and \([\bar{c}, d]\) as well as between the part in \([\bar{c}, d]\) and the end point \(\frac{1}{q^4 n_i-1}\) of \(I_{n_i}\). Then similar to Lemma 4.1, we have
\[
\left|\frac{d^r \bar{h}_i(x)}{dx^r}\right| \leq C(r) \cdot q_{n_i}^{3r}, \quad 0 \leq r \leq \left[\frac{1}{q^4 n_i}\right].
\]
Thus the \(C^\infty\)-convergence of \(C_k(x)\) is similar to the above argument. The remain part of the proof is same as Section 3.

## A Product of hyperbolic matrices

Let \(A\) be a hyperbolic \(SL(2, R)\)-matrix, i.e., \(\|A\| > 1\). It is know that \(A\) can be written uniquely as \(A = R_\psi \cdot \Lambda_A \cdot R_\phi\) with \(\Lambda_A = \text{diag}(\|A\|, \|A\|^{-1})\). It is known that \(-\phi\) is the most expanded direction of \(A\) and \(\psi\) is the most contracted direction of \(A^{-1}\).

For two hyperbolic matrices \(A = R_\psi A \cdot \Lambda_A \cdot R_\psi, B = R_\psi B \cdot \Lambda_B \cdot R_\psi B\) with big norms, let \(BA = R_\psi BA \cdot \Lambda_{BA} \cdot R_\psi BA\). We firstly investigate how \(\phi_{BA}, \psi_{BA}\) and \(\|BA\|\) depend on \(A\) and \(B\).

**Lemma A.1.** Let \(A, B\) be hyperbolic \(SL(2, \mathbb{R})\) cocycles and \(\theta = \phi_B + \psi_A\). Then it holds that
\[
\frac{1}{4} N(\|A\|, \|B\|, \theta) \leq \|BA\|^2 \leq N(\|A\|, \|B\|, \theta),
\]
where
\[
N(\|A\|, \|B\|, \theta) = (\|A\|^2 \|B\|^2 + \|A\|^{-2} \|B\|^{-2}) \cdot \cos^2 \theta + (\|A\|^2 \|B\|^{-2} + \|A\|^{-2} \|B\|^2) \cdot \sin^2 \theta.
\]

**Proof.** For any \(SL(2, \mathbb{R})\) matrix \(A = (a_{ij})_{2\times 2}\), it is known that
\[
\frac{1}{4} \sum_{i,j} a_{ij}^2 \leq \|A\|^2 \leq \sum_{i,j} a_{ij}^2.
\]

It is easy to see that
\[
\|BA\| = \left\| \begin{pmatrix} \|B\| & 0 \\ 0 & \|B\|^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} \|A\| & 0 \\ 0 & \|A\|^{-1} \end{pmatrix} \right\| = \left\| \begin{pmatrix} \|A\| \|B\| \cos \theta & -\|A\|^{-1} \|B\| \sin \theta \\ \|A\| \|B\|^{-1} \sin \theta & \|A\|^{-1} \|B\|^{-1} \cos \theta \end{pmatrix} \right\|.
\]

It thus implies the conclusion. \(\square\)
Lemma A.2. Let \( \phi = \phi_A - \phi_{BA}, \psi = \psi_{BA} - \psi_B \). Assume \( \theta \in [0, \pi) \). Then

\[
\phi(||A||, ||B||, \theta) = \begin{cases}
0, & \text{for } \theta = 0 \\
-\frac{1}{2} \left( \frac{\pi}{2} - \tan^{-1}(a \cot \theta + b \tan \theta) \right), & \text{for } 0 < \theta < \frac{\pi}{2} \\
\frac{\pi}{2} - \frac{1}{2} \left( \frac{\pi}{2} - \tan^{-1}(a \cot \theta + b \tan \theta) \right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b \geq 0 \\
-\frac{\pi}{2} - \frac{1}{2} \left( \frac{\pi}{2} - \tan^{-1}(a \cot \theta + b \tan \theta) \right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b < 0 \\
0, & \text{for } \theta = \frac{\pi}{2} \text{ if } b \geq 0 \\
-\frac{\pi}{2}, & \text{for } \theta = \frac{\pi}{2} \text{ if } b < 0,
\end{cases}
\]

where

\[
a = \frac{||A||^2 ||B||^2 - ||A||^{-2} ||B||^{-2}}{2(||B||^2 - ||B||^{-2})}, \quad b = \frac{||A||^2 ||B||^{-2} - ||A||^{-2} ||B||^2}{2(||B||^2 - ||B||^{-2})}.
\]

Similarly,

\[
\psi(||A||, ||B||, \theta) = \begin{cases}
0, & \text{for } \theta = 0 \\
-\frac{1}{2} \left( \frac{\pi}{2} - \tan^{-1}(a' \cot \theta - b' \tan \theta) \right), & \text{for } 0 < \theta < \frac{\pi}{2} \\
\frac{\pi}{2} - \frac{1}{2} \left( \frac{\pi}{2} - \tan^{-1}(a' \cot \theta - b' \tan \theta) \right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b' \geq 0 \\
-\frac{\pi}{2} - \frac{1}{2} \left( \frac{\pi}{2} - \tan^{-1}(a' \cot \theta - b' \tan \theta) \right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b' < 0 \\
0, & \text{for } \theta = \frac{\pi}{2} \text{ if } b' \geq 0 \\
-\frac{\pi}{2}, & \text{for } \theta = \frac{\pi}{2} \text{ if } b' < 0,
\end{cases}
\]

where

\[
a' = \frac{||A||^2 ||B||^2 - ||A||^{-2} ||B||^{-2}}{2(||A||^2 - ||A||^{-2})}, \quad b' = \frac{||A||^2 ||B||^{-2} - ||A||^{-2} ||B||^2}{2(||A||^2 - ||A||^{-2})}.
\]

Proof. Let

\[
V(s) = \begin{pmatrix} ||B|| & 0 \\ 0 & ||B||^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} ||A|| & 0 \\ 0 & ||A||^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}
\]

\[
= \begin{pmatrix} ||B|| & 0 \\ 0 & ||B||^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos \theta \cdot ||A|| \cdot \cos s - \sin \theta \cdot ||A||^{-1} \cdot \sin s \\ \sin \theta \cdot ||A|| \cdot \cos s + \cos \theta \cdot ||A||^{-1} \cdot \sin s \end{pmatrix}
\]

\[
= \begin{pmatrix} \cos \theta \cdot ||A|| \cdot ||B|| \cdot \cos s - \sin \theta \cdot ||A||^{-1} \cdot ||B|| \cdot \sin s \\ \sin \theta \cdot ||A|| \cdot ||B||^{-1} \cdot \cos s + \cos \theta \cdot ||A||^{-1} \cdot ||B||^{-1} \cdot \sin s \end{pmatrix}.
\]
Thus
\[ |V(s)|^2 = (\cos \theta \|A\| \|B\|)^2 + (\sin^2 \theta \|A\|^2 \|B\|^2 - \cos^2 \theta \|A\|^2 \|B\|^2) \sin^2 s + \sin^2 \theta \|A\|^2 \|B\|^2 \]
\[ + (\cos^2 \theta \|A\|^2 \|B\|^2 - \sin^2 \theta \|A\|^2 \|B\|^2) \sin^2 s + 2(\|B\|^2 - \|B\|^2) \sin \theta \cos \theta \sin s \cos s. \]

Obviously \( \frac{d}{ds}(|V(s)|^2) = 0 \) at \( \phi \) since \( |V(s)|^2 \) attains its extreme at \( \phi \), a simple computation leads to
\[
\left( (\|A\|^2 B^2 - \|A\|^2 \|B\|^2) \cos^2 \theta + (\|A\|^2 \|B\|^2 - \|A\|^2 \|B\|^2) \sin^2 \theta \right) \sin 2\phi
= -2(\|B\|^2 - \|B\|^2) \sin 2\theta \cos 2\phi.
\]
Thus
\[ -\cot 2\phi = \frac{\|A\|^2 \|B\|^2 - \|A\|^2 \|B\|^2}{2(\|B\|^2 - \|B\|^2)} \cot \theta + \frac{\|A\|^2 \|B\|^2 - \|A\|^2 \|B\|^2}{2(\|B\|^2 - \|B\|^2)} \tan \theta. \]

With the help of the inequality \( \frac{d^2}{ds^2}(|V(s)|^2) \leq 0 \), we obtain the unique \( \phi \) corresponding the maximum \( \|BA\|^2 \) of \( |V(s)|^2 \), which satisfies (A.16).

(A.17) is proved similarly. \( \square \)

Later we will see that both \( \|A\| \) and \( \|B\| \) are very big. Thus
\[ a = \frac{\|A\|^2 \|B\|^2 - \|A\|^2 \|B\|^2}{\|B\|^2 - \|B\|^2} \sim \|A\|^2, \]
\[ b = \frac{\|A\|^2 \|B\|^2 - \|A\|^2 \|B\|^2}{\|B\|^2 - \|B\|^2} \lesssim \max\{\|A\|^2, \frac{\|A\|^2}{\|B\|^2}\}. \]

If \( A, B \) are hyperbolic, the functions \( \phi(\|A\|, \|B\|, \theta), \psi(\|A\|, \|B\|, \theta) \) defined above are continuous in all variables. In the following, we estimate the derivatives of \( \phi \) and \( \psi \) with respect to \( \theta \), \( \|A\| \) and \( \|B\| \).

Lemma A.3. It holds that
\[ |\phi(\text{mod } \pi)| \leq C(0) \cdot \|A\|^2 \cdot |\theta - \frac{\pi}{2}|^{-1} \quad (A.18) \]
and
\[ |\psi(\text{mod } \pi)| \leq C(0) \cdot \|B\|^2 \cdot |\theta - \frac{\pi}{2}|^{-1}. \quad (A.19) \]
Suppose \( |\theta - \frac{\pi}{2}|^{-1} \ll \|A\|^2 \). Then, for \( i \geq 1 \), we have that
\[ \left| \frac{\partial^i \phi}{\partial \theta^i} \right| \leq C(i) \cdot \|A\|^2 \cdot |\theta - \frac{\pi}{2}|^{-i-1}, \quad (A.20) \]
\[ \left| \frac{\partial^i \phi}{\partial \|A\|^i} \right| \leq C(i) \cdot \|A\|^2 \cdot \|A\|^{-i} \cdot |\theta - \frac{\pi}{2}|^{-1}, \quad (A.21) \]
and
\[ \left| \frac{\partial^i \phi}{\partial \|B\|^i} \right| \leq C(i) \cdot \|A\|^2 \cdot \|B\|^{-i} \cdot |\theta - \frac{\pi}{2}|^{-1}. \quad (A.22) \]
More generally, for $i + j + k \geq 1$, we have

$$|\frac{\partial^{i+j+k} \phi}{\partial \theta^i \partial A^j \partial B^k}| \leq C(i, j, k) \cdot |\theta - \frac{\pi}{2}|^{-i-1} \cdot \|A\|^{-2-j} \cdot \|B\|^{-k};$$  \hspace{1cm} (A.23)

Similarly, suppose $|\theta - \frac{\pi}{2}|^{-1} \ll \|B\|^2$. Then we have

$$|\frac{\partial^{i+j+k} \phi}{\partial \theta^i \partial A^j \partial B^k}| \leq C(i, j, k) \cdot |\theta - \frac{\pi}{2}|^{-i-1} \cdot \|A\|^{-j} \cdot \|B\|^{-2-k}.$$  \hspace{1cm} (A.24)

**Proof.** To prove (A.18), it is sufficient to consider the situation $\theta \approx \frac{\pi}{2}$. We only consider the case $0 \leq \theta \leq \frac{\pi}{2}$ since the proof for the other cases is similar. From the fact $\lim_{x \to \infty} \frac{\pi - \arctan x}{x} = 1$ and the definition of $a$, we have $|\phi| \leq C(0) \cdot a^{-1} \cdot |\theta - \frac{\pi}{2}|^{-1} \leq C(0) \cdot \|A\|^{-2} \cdot |\theta - \frac{\pi}{2}|^{-1}$. Thus we obtain (A.18). We can obtain (A.19) similarly.

For $i \geq 1$, from the definition of $\phi$, we have

$$\frac{\partial^{i} \phi}{\partial \theta^i} = -\frac{1}{2} \sum_{l_1 + \cdots + l_k = i} \frac{d^{k-1} f}{df^{k-1}} (\frac{1}{1+f^2}) \cdot \frac{\partial^{l_1} f}{\partial \theta^{l_1}} \cdots \frac{\partial^{l_k} f}{\partial \theta^{l_k}},$$

where $f(\|A\|, \|B\|, \theta) = a \cot \theta + b \tan \theta$. To estimate $\frac{\partial^{i} \phi}{\partial \theta^i}$, we have that

$$|\frac{\partial^{l_k} f}{\partial \theta^{l_k}}| = |\frac{\partial^{l_k} f}{\partial \theta^{l_k}} (a \cot \theta + b \tan \theta)| \leq |a| \cdot |\cot^{(l_k)}(\theta)| + |b| \cdot |\tan^{(l_k)}(\theta)|.$$

By a direct computation, we have

$$|\tan^{(l_k)}(\theta)| = |(\cos^{-2} \theta)^{(l_k-1)}| \leq |\sum_{\kappa_1 + \cdots + \kappa_t = l_k-1} \cos^{-2+\kappa_1} \theta \cdot \cos^{\kappa_1} \theta \cdots \cos^{\kappa_t} \theta|$$

and

$$|\cot^{(l_k)}(\theta)| = |(\sin^{-2} \theta)^{(l_k-1)}| \leq |\sum_{\kappa_1 + \cdots + \kappa_t = l_k-1} \sin^{-2+\kappa_1} \theta \cdot \sin^{\kappa_1} \theta \cdots \sin^{\kappa_t} \theta|.$$

From the condition $|\theta - \frac{\pi}{2}|^{-1} \ll \|A\|^2$ and the fact that the signs of $\|B\|^2 \cot \theta$ and $\|B\|^2 \tan \theta$ are the same, we have

$$|\frac{\partial^{l_k} f}{\partial \theta^{l_k}}| \leq C(l_k) \cdot (|a| \cdot |\theta - \frac{\pi}{2}|^{-l_k-1} + |b| \cdot |\theta - \frac{\pi}{2}|^{-l_k-1}) \leq C(l_k) \cdot |f| \cdot |\theta - \frac{\pi}{2}|^{-l_k}.$$  \hspace{1cm} (A.25)

On the other hand, we have

$$|\frac{d^{k-1} f}{df^{k-1}} (\frac{1}{1+f^2})| \lesssim |f|^{-k-1} \text{ if } k \geq 1.$$

Thus from $|f| \gtrsim \|A\|^2 \cdot |\cot \theta|$ we obtain

$$|\frac{\partial^{i} \phi}{\partial \theta^{i}}| \leq C(i) |\theta - \frac{\pi}{2}|^{-i} \cdot \frac{1}{|f|} \leq C(i) \|A\|^{-2} |\theta - \frac{\pi}{2}|^{-i-1}.$$
Next we estimate
\[
\left| \frac{\partial^k \phi}{\partial |A|^k} \right| \leq \sum_{l_1 + \cdots + l_k = i} \left| \frac{d^{k-1} (\frac{\partial}{\partial |A|^k})}{d|A|^k} \right| \left| \frac{\partial^{l_j} f}{\partial |A|^{l_j}} \right| \cdots \left| \frac{\partial^{l_k} f}{\partial |A|^{l_k}} \right| \quad l_j \geq 1, 1 \leq j \leq k
\] (A.26)
\[
\leq \sum_{l_1 + \cdots + l_k = i} \left| \frac{\partial^{l_j} f}{\partial |A|^{l_j}} \right| \cdots \left| \frac{\partial^{l_k} f}{\partial |A|^{l_k}} \right|.
\]

It is easy to see that \( |f| \sim |a| \cdot |\cot \theta| \) with the condition \( |\theta - \frac{\pi}{2}|^{-1} \ll |A|^2 \). We also have
\[
\left| \frac{\partial^k a}{\partial |A|^k} \right| \leq |\cot \theta| \left| \frac{\partial^k a}{\partial |A|^k} \right| + |\tan \theta| \left| \frac{\partial^k b}{\partial |A|^k} \right|.
\]
By a direct computation, we obtain
\[
\left| \frac{\partial^k a}{\partial |A|^k} \right| = \left| \frac{\partial^k a}{\partial |A|^k} \right| \leq C(l_s) \cdot |a| \cdot |A|^{-l_s}
\]
and
\[
\left| \frac{\partial^k b}{\partial |A|^k} \right| = \left| \frac{\partial^k b}{\partial |A|^k} \right| \leq C(l_s) \cdot \left( |A|^{-2} + \frac{|A|^2}{|B|^2} \right) \cdot |A|^{-l_s}.
\]
Thus we have
\[
\left| \frac{\partial^k a}{\partial |A|^k} \right| \leq C(l_s) \cdot \left\{ \left| \theta - \frac{\pi}{2} \right| |A|^{-2-l_s} + |\theta - \frac{\pi}{2}|^{-1} \cdot |A|^{-l_s} \cdot \left( |A|^{-2} + \frac{|A|^2}{|B|^2} \right) \right\}
\] (A.27)
\[
\leq C(l_s) \cdot |f| \cdot |A|^{-l_s}.
\]
With the fact that \( |f| \gtrsim |A|^2 \cdot |\cot \theta| \), it follows that
\[
\left| f \right|^2 \cdot \left| \frac{\partial^k a}{\partial |A|^k} \right| \leq C(l_s) \cdot |f|^{-1} \cdot |A|^{-l_s}
\] (A.28)
\[
\leq C(l_s) \cdot |A|^{-2-l_s} \cdot \left| \theta - \frac{\pi}{2} \right|^{-1}.
\]
Combining (A.26), (A.27) with (A.28), we obtain (A.21). Similarly, we have (A.22) and (A.23). The estimates for \( \psi \) can be proved similarly. \( \square \)

## B Proof of Lemma 2.1

In this section, we first give estimates on most contracted and expanded directions of the product of hyperbolic blocks. Then we will give the proof of Lemma 2.1.

Let \( A(x), B(x), \theta(x), \phi(x) \) and \( \psi(x) \) be defined as in Lemma A.1 and A.2.

**Lemma B.1.** Let \( |\theta - \frac{\pi}{2}|^{-1} \ll |A|^2, |B|^2 \). Suppose that, for any \( i \geq 0 \),
\[
\left| \frac{d^i |A|}{dx^i} \right| \leq C(i) \cdot |A| \cdot |\theta - \frac{\pi}{2}|^{-i-1}, \quad \left| \frac{d^i |B|}{dx^i} \right| \leq C(i) \cdot |B| \cdot |\theta - \frac{\pi}{2}|^{-i-1}, \quad \left| \frac{d^i \theta}{dx^i} \right| \leq C(i) \cdot |\theta - \frac{\pi}{2}|^{-i-1}.
\] (B.29)
Then we have
\[
\left| \frac{d^i \phi}{dx^i} \right| \leq C(i) \cdot |\theta - \frac{\pi}{2}|^{-i-1} \cdot \|A\|^{-2},
\]
\[
\left| \frac{d^i \psi}{dx^i} \right| \leq C(i) \cdot |\theta - \frac{\pi}{2}|^{-i-1} \cdot \|B\|^{-2},
\]
\[
\left| \frac{d^i \|BA\|}{dx^i} \right| \leq C(i) \cdot \|BA\| \cdot |\theta - \frac{\pi}{2}|^{-i-1}.
\] (B.30)

Proof. From Lemma A.3 and
\[
\frac{d^i \phi}{dx^i} = \sum_{t_1 + \cdots + t_i = i} \sum_{s_1 + \cdots + s_i = 1} \left| \frac{\partial^{i+1+2+3} \phi}{\partial |A|^{i} \cdot \theta \cdot |B|^{2} \cdot \partial \theta^{3}} \right| \cdot \frac{\partial^i \|A\|}{dx^i} \cdot \frac{\partial^i \|A\|}{dx^i},
\]
we have
\[
\left| \frac{\partial^{i+1+2+3} \phi}{\partial |A|^{i} \cdot \theta \cdot |B|^{2} \cdot \partial \theta^{3}} \right| \leq C(i_1, i_2, i_3) \cdot |\theta - \frac{\pi}{2}|^{-i_1-i_2-i_3} \cdot \|A\|^{-i_1} \cdot \|B\|^{-i_2}.
\] (B.32)

Then from (B.29), (B.31), (B.32), we have
\[
\left| \frac{d^i \phi}{dx^i} \right| \leq C(i) \cdot \|A\|^{-2} \cdot |\theta - \frac{\pi}{2}|^{-i_1-i_2-i_3},
\]
thus the first inequality of (B.30) is proved. In the above inequality, we used the fact
\[
\left| \frac{\partial f}{\partial \|A\|} \cdot \frac{\partial^i \|A\|}{dx^i} \right| \leq |\theta - \frac{\pi}{2}|^{-i_1-i_2-i_3} \cdot |f|.
\]

The second inequality is proved similarly. Now we prove the third inequality. By a direct computation, we have
\[
\frac{\partial^i \|BA\|}{\partial \phi^i} = \frac{\partial^i (g^2)}{\partial \phi^i} = \sum_{t_1 + \cdots + t_i = i} (g^2)^{(i)} \cdot \frac{\partial^i g}{\partial \phi^i} \cdot \frac{\partial^i g}{\partial \phi^i},
\] (B.33)
where
\[
g = g_1^2 + g_2^2, \quad g_1 = \|A\| \cdot \|B\| \cdot \cos \theta \cos \phi - \|A\|^{-1} \cdot \|B\|^{-1} \sin \theta \sin \phi,
\]
\[
g_2 = \|A\| \cdot \|B\|^{-1} \cdot \sin \theta \cdot \cos \phi + \|A\|^{-1} \cdot \|B\|^{-1} \cdot \cos \theta \cdot \sin \phi.
\] (B.34)

It is not difficult to see that \(|(g^2)^{(i)}| \leq C(k) \cdot g^{\frac{1}{2} - k} \). From the definition of \(g\), we have
\[
\left| \frac{\partial^i g}{\partial \phi^i} \right| \leq \left| \frac{\partial^i (g^2_1)}{\partial \phi^i} \right| + \left| \frac{\partial^i (g^2_2)}{\partial \phi^i} \right|,
\]
From
\[
\left| \frac{\partial^i (g^2_1)}{\partial \phi^i} \right| \leq \sum_{t_1 + t_2 = i} \left| \frac{\partial^i g_1}{\partial \phi^i} \right| \cdot \left| \frac{\partial^i g_2}{\partial \phi^i} \right|,
\]
\[
\left| \frac{\partial^i (g^2_2)}{\partial \phi^i} \right| \leq \sum_{t_1 + t_2 = i} \left| \frac{\partial^i g_1}{\partial \phi^i} \right| \cdot \left| \frac{\partial^i g_2}{\partial \phi^i} \right|,
\]

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it follows that
\[
\left| \frac{\partial^k (g^2)}{\partial \psi^i} \right| \leq \|A\| \|B\| \|\cos \theta\| \cdot |\cos (\phi + \frac{\pi}{2} \cdot l_{s,k})| + \|A\|^{-1} \|B\| \|\sin \theta\| \cdot |\sin (\phi + \frac{\pi}{2} \cdot l_{s,k})|
\]
\[
\leq \|A\| \cdot \|B\| \cdot |\cos \theta| \leq \|BA\|.
\]

Then we obtain
\[
\left| \frac{\partial^k (g^2)}{\partial \psi^i} \right| \leq \|BA\|^2. 
\]  
(B.35)

Similarly, we have
\[
\left| \frac{\partial^k (g^2)}{\partial \theta^i} \right| \leq \|BA\|^2. 
\]  
(B.36)

Combining (B.33) with (B.35), (B.36), we then have
\[
\left| \frac{\partial^k \|BA\|}{\partial \phi^i} \right| \leq C(i) \cdot \max_{k \leq i} (\|BA\|^{2k} \cdot g^{\frac{1}{2} - k}) = C(i) \cdot \|BA\|. 
\]  
(B.37)

Similarly, it holds that
\[
\left| \frac{\partial^k \|BA\|}{\partial \theta^i} \right| \leq C(i) \cdot \max_{k \leq i} \left( g^{\frac{1}{2} - k} \cdot (\|A\| \|B\|)^{2k} \right) \|\cos \theta|^k \right) \leq C(i) \cdot |\theta - \frac{\pi}{2}|^{1-i} 
\]  
(B.38)

and
\[
\left| \frac{\partial^k \|BA\|}{\partial \|A\|^i} \right| \leq C(i) \cdot \|BA\| \cdot \|A\|^{-i}, \quad \left| \frac{\partial^k \|BA\|}{\partial \|B\|^i} \right| \leq C(i) \cdot \|BA\| \cdot \|B\|^{-i}. 
\]  
(B.39)

Similar to (B.37)-(B.39), we have
\[
\left| \frac{\partial^{i_1 + \cdots + i_4} \|BA\|}{\partial \|A\|^{i_1} \cdot \partial \|B\|^{i_2} \cdot \partial^4 \theta} \right| \leq C(i) \cdot \|BA\| \cdot \|A\|^{-i_1} \cdot \|B\|^{-i_2} \cdot |\theta - \frac{\pi}{2}|^{-i_4}.
\]

Combining with (B.1), the first inequality in (B.30) and the fact
\[
\frac{d^i \|BA\|}{dx^i} = \sum \frac{\partial^{i_1 + \cdots + i_4} \|BA\|}{\partial \|A\|^{i_1} \cdot \partial \|B\|^{i_2} \cdot \partial^4 \theta} \cdot \frac{\partial^{i_1,1} \|A\|}{\partial x^{j_1,1}} \cdot \frac{\partial^{i_1,2} \|A\|}{\partial x^{j_2,1}} \cdot \frac{\partial^{i_1,3} \|A\|}{\partial x^{j_3,1}} \cdot \frac{\partial^{i_1,4} \|A\|}{\partial x^{j_4,1}},
\]
we prove the third inequality in (B.30).  \(\Box\)

**Proof of Lemma 2.1.** For any \(x \in I_{k+1}\), let \(r_k(x) := r_{k,0}(x) < r_{k,1}(x) < \cdots < r_{k,s(x)}(x) := r_{k+1}(x)\) such that \(T^{r_{k,j}(x)} x \in I_k, 0 \leq j \leq s(x) \leq C(M)\). Consider \(A^{r_{k,0}(x)+r_{k,1}(x)}(x) = A^{r_{k,1}(x)}(T^{r_{k,0}(x)}(x)) \cdot A^{r_{k,0}(x)}(x)\).

Let \(A^{r_{k,0}(x)}(x) := R_{\psi_k^{-}}(x) \cdot \Lambda_k^{-}(x) \cdot R_{\phi_k^{-}}(x)\) and \(A^{r_{k,1}(x)}(T^{r_{k,0}(x)}(x)) := R_{\psi_k^{+}}(x) \cdot \Lambda_k^{+}(x) \cdot R_{\phi_k^{+}}(x)\). Then
\[
A^{r_{k,0}(x)+r_{k,1}(x)}(x) = R_{\psi_k^{+}}(x) \cdot R_{\psi_k^{-}}(x) \cdot \Lambda_k^{+}(x) \cdot \Lambda_k^{-}(x) \cdot R_{\phi_k^{-}}(x) := R_{\psi_{k+1,1}^{-}}(x) \cdot \Lambda_{k+1,1}(x) \cdot R_{\phi_{k+1,1}^{-}}(x).
\]
Since $T^{r_{k,j}(x)}x \in I_k \setminus I_{k+1}$ for $j < s(x)$, it holds that $|\phi^+_k + \psi^-_k - \frac{\pi}{2}| \geq d_{k+1}$. From (2.7) and Lemma B.1, we have

$$\left| \frac{d^i(\phi_{k+1,1} - \phi_k^-)}{dx^i} \right| \leq C(i) \cdot d_{k+1}^{-i} \cdot \|A^{r_{k,1}(x)}(x)\|^{-2}. \quad (B.40)$$

Similarly, it holds that

$$\left| \frac{d^i(\psi_{k+1,1} - \psi_k^+)}{dx^i} \right| \leq C(i) \cdot d_{k+1}^{-i} \cdot \|A^{r_{k,0}(x)}(x)\|^{-2}. \quad (B.41)$$

In the above, we regard $\phi_{k+1,1} - \phi_k^-$ and $\phi_k^+ + \psi_k^-$ as $\phi$ and $\theta$ in Lemma B.1, respectively. Moreover, for each $x \in I_k$, $|\theta(x) - \frac{\pi}{2}| = |\phi_k^+(x) + \psi_k^-(x) - \frac{\pi}{2}| \geq d_{k+1}$ from the definition of $d_{k+1}$. It implies that

$$\left| \frac{d^i\phi_{k+1,1}}{dx^i} \right|, \quad \left| \frac{d^i\psi_{k+1,1}}{dx^i} \right| \leq C(i) \cdot d_{k+1}^{-i} + \left| \frac{d^i\phi_k^-}{dx^i} \right| + \left| \frac{d^i\psi_k^+}{dx^i} \right| \leq C(i) \cdot (d_{k+1}^{-i} + d_{k+1}^{-i}) \leq C(i) \cdot d_{k+1}^{-i}.$$

The last inequality is obtained from (1)\textsuperscript{k}.

From Lemma B.1 we obtain

$$\left| \frac{d^i\|A^{r_{k,0}(x)} + r_{k,1}(x)\|}{dx^i} \right| \leq C(i) \cdot \|A^{r_{k,0}(x)} + r_{k,1}(x)\| \cdot d_{k+1}^{-i}.$$\textsuperscript{3}

Since $s(x) \leq C(M)$, it follows from no more than $C(M)$-applications of the above argument that (1)\textsubscript{k+1} and (2)\textsubscript{k+1} hold true.

Iterating (B.40) and (B.41) less than $C(M)$ times, we get

$$\left| \frac{d^i}{dx^i}(\phi_{A^{r_{k+1}}(x)} - \phi_{A^{r_k^+}}(x)) \right| \leq C(i) \cdot |\theta_k - \frac{\pi}{2}|^{-i} \cdot \|A^{r_k^+}\|^{-2}$$

and

$$\left| \frac{d^i}{dx^i}(\psi_{A^{r_{k+1}}}(x) - \psi_{A^{r_k^-}}(x)) \right| \leq C(i) \cdot |\theta_k - \frac{\pi}{2}|^{-i} \cdot \|A^{r_k^-}\|^{-2}.$$\textsuperscript{4}

Since $|\theta_k - \frac{\pi}{2}| \geq d_k$, we obtain (2.8).

\[ \square \]

Remark B.1. In the proof of Lemma 2.1, it is not necessary that $s(x)$ is bounded by a constant. We make such an assumption only for the simplicity. And the condition that $\omega$ is of bound type is only used in constructing $C_k(x)$.

Acknowledgments. The authors would like to thank S. Jitomirskaya for drawing their attention to the problem. Y.Wang was supported by NSFC (grant no.11271183), J. You was supported by NSFC (grant no. 11471155) and 973 projects of China (grant no. 2014CB340701).
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