BLACK-SCHOLES EQUATION FOR ILLIQUID MARKET*

CARLOS OLIVEIRA † AND MANUEL GUERRA ‡

Abstract. Perfect liquidity of risky assets is a strong assumption in the Black-Scholes model. Several authors proposed alternative models accounting for imperfect liquidity. One such model was proposed and extensively discussed by Schönbucher and Wilmott (2000).

In the present contribution, we argue that the definition of self-financed strategy used in that paper does not take into full account the effects of imperfect liquidity introduced in the model. We propose one alternative formulation and discuss some properties of the resulting price process.

We use the modified model to discuss the effect of collective behaviour by a large number of small hedgers. If a large number of small traders use similar strategies wrongly assuming perfect liquidity, then synchronized trading of large quantities may have a significant impact in the strategy outcome.

We show that in such circumstances, the expected outcome of the classical Black-Scholes hedging strategy for an European put option can diverge significantly from the perfect hedging obtained under perfect liquidity. The effect of illiquidity can be described by a nonlinear Black-Scholes equation having some very unusual features.

Key words. Black-Scholes Equations, Illiquid Markets, Option pricing.

AMS subject classifications. 90A09, 90A60

1. Introduction. The Black-Scholes model is used to price and to design hedging strategies in different securities. Black and Scholes in [1] and Merton in [5] simultaneously derived the Black-Scholes formula to price European options.

When the financial market is competitive and complete it is possible to build a portfolio with risky asset and cash that mimics the path of the price process of a contingent claim allowing to replicate this derivative. In finance, this argument is used in order to obtain the option pricing equations. However, in this paper we use the approach of the stochastic optimal control and Hamilton-Jacobi-Bellman (HJB) (see for example Frey and Polte [2]) to derive a Black-Scholes equation. This strategies can be used to hedge its exposure or to increasing its profit for an arbitrager.

This theory is largely used in the financial industry, but requires strong assumptions. Usually, it is assumed that the market is competitive and all agents are price takers. This means that any individual trading in the risky asset does not influence significantly its price. This is a reasonable approximation of reality when the market contains a large number of tratders of similar size and the aggregate quantity traded in every moment far exceeds the holdings of any individual. However, in financial markets there are some companies, funds and other institutions of such large dimension that the effects of their actions on the asset prices are not negligible.

Oddly, there are comparatively few models in the literature that relax this assumption. Due to their dimension, large traders can manipulate market prices to their profit, and models taking this into account are hurt by theoretical difficulties that are discussed, among others, by Wilmott and Schönbucher in [6] and Jarrow in [3] and [4].

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Wilmott and Schönbucher in [6] proposed a simple approach to model the dynamics of prices in illiquid financial markets and to discuss its consequences in the dynamic trading strategies. In this paper, we show that the self-financing condition used in [6] does not take into full account the feedback effects due to hedging by a large trader. We use the Schönbucher-Wilmott model to discuss the effect of synchronized behaviour by hedgers using similar strategies. We obatin a Black-Scholes equation for the true value of their hedging portfolio.

In section 2 we briefly discuss the model proposed by Wilmott and Schönbucher [6]. In particular, we propose an alternative self-financing condition taking into full account the feedback effects of hedging by a large trader, and describe the resulting price dynamics in an illiquid market. In Section 3 we obtain the Black-Scholes equation for the true value of a delta-hedging strategy by synchronized traders that are unaware of their collective behaviour.

2. The Wilmott-Schönbucher model. In the literature there are comparatively few models taking into account the illiquidity of the risky asset market. Here we present summarily a model proposed by Wilmott and Schönbucher in [6]. We discuss the concept of portfolio value as well and we propose a new definition of self-finance strategy for this market.

2.1. The price mechanism. The model considers two types of assets, a risky one, with price S and a risk-free one, with price B. The risk-free asset is taken as numeraire with B_0 normalized to 1. The market of the risk-free asset is perfectly liquid but the market of the risky asset is not. There are two types of agents in the market: A single large trader and a large set of small traders.

The aggregate demand of the risky asset by the small traders at time t is a function $D(S, W, t)$, where S denotes the price of the risky asset and W is a random parameter. Similarly, the aggregate supply by the small traders is a function $Su(S, W, t)$. All information that arrives to small traders is contained in W. Thus, the small traders don't have any knowledge about the presence of the large trader in the market. The excess demand is by definition the difference between demand and supply,

$$\chi(S, W, t) = D(S, W, t) - Su(S, W, t).$$

In the absence of the large trader, the equilibrium price at time t is the solution of the equilibrium equation

$$\chi(S, W_t, t) = 0. \tag{2.1}$$

Assuming $\chi(S, W, t)$ is smooth and

$$\frac{\partial \chi}{\partial S}(S, W, t) < 0, \forall (S, W, t) \in]0, +\infty[\times \mathbb{R} \times [0, +\infty[, \tag{2.2}$$

the equilibrium price is the unique $C^{2,1}$-function, $S = \Phi(W, t)$ implicitly defined by (2.1). Economically, the condition (2.2) means that when the price goes up the excess of demand goes down, as occurs when supply increases and demand decreases with price.

Let f denote the quantity of risky asset held by the large trader. It is assumed that this quantity is draw from the aggregate quantity available in the market. Therefore, the equilibrium equation (2.1) becomes

$$\chi(S, W, t) + f = 0. \tag{2.3}$$

This condition defines the equilibrium price in the presence of the large trader as an implicit $C^{2,2,1}$-function, $S = \Psi(f, W, t)$, provided χ is sufficiently smooth and (2.2) holds.

Assuming that f_t, the quantity held by the large trader at time t, and $W = W_t$ are càdlàg process, then the price $S_t = \Psi(f_t, W_t, t)$ is also a càdlàg process.

In the following we assume that W_t is a standard Brownian motion.

2.2. The effect of large transactions and the large trader's self-financing strategies. Suppose that at time t the large trader wants to change his hold of risky asset from the quantity f_{t^-} to the quantity f_t. Taking this transaction into account, the asset price at time t becomes $S_t = \Psi(f_t, W_t, t)$. In the absence of this transaction it would be $\tilde{S}_t = \Psi(f_{t^-}, W_t, t)$. Thus, we need to consider carefully how to compute the value of such a transaction.

To do this, we assume that a transaction at time t is made with the knowledge of W_t. This is consistent with the self-financing condition in the perfectly liquid market

$$(f_t - f_{t^-})\, S_t + (c_t - c_{t^-})\, B_t = 0,$$

where c_t denotes the quantity of bonds held at time t. Under this assumption, continuity of Ψ implies that any price between $\Psi(f_t, W_t, t)$ and $\Psi(f_{t^-}, W_t, t)$ is acceptable for some trader in the market. Since the large trader seeks the best possible bargain, we assume that he gives priority to higher bidders and lower askers. Therefore, by selling sequentially from higher to lower bidders (buying sequentially from lower to higher askers) at their respective prices, the value of the transaction is $\int_{f_{t-}}^{f_t} \Psi(x, W_t, t)dx$. Therefore, if the larger trader's strategy is self-financed, then the the variation in the hold of riskless asset $c_t - c_{t^-}$ must satisfy

$$\int_{f_{t-}}^{f_t} \Psi(x, W_t, t)dx + (c_t - c_{t-})B_t = 0. \tag{2.4}$$

If the process f_t is smooth, then a similar argument leads to the usual self-financing condition, also used in [6]:

$$S_t df_t + B_t dc_t = 0.$$

Below we show that, in general, we should not expect the process f_t to be smooth and that this implies that a different self-financing condition must be considered.

2.3. Feedback effects. From the economical point of view, one expects that the quantity of risky asset held by the large trader should depend on its price. Thus, Schönbucher and Wilmott [6] assume the process f_t to be of the form $f_t = f(S_t, t)$, where $f(\cdot, \cdot)$ is a smooth function.

In this case, the equilibrium condition (2.3) becomes

$$\chi(S, W, t) + f(S, t) = 0. \tag{2.5}$$

Assuming that the analogous of condition (2.2)

$$\frac{\partial \chi}{\partial S}(S, W, t) + \frac{\partial f}{\partial S}(S, t) < 0, \forall (S, W, t) \in]0, +\infty[\times \mathbb{R} \times [0, +\infty[\tag{2.6}$$

holds, then (2.5) defines the equilibrium price as an implicit $C^{2,1}$-function, $S = \Psi_1(W, t)$.

Therefore, the price process $S_t = \Psi_1(W_t, t)$ is a continuous nonsmooth process and the same holds for the process $f_t = f(S_t, t) = f(\Psi_1(W_t, t), t)$.

PROPOSITION 2.1. *For the illiquid market, we have*

$$dS_t = \left(\frac{\partial \Psi_1}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi_1}{\partial w^2}\right) dt + \frac{\partial \Psi_1}{\partial w} dW_t, \tag{2.7}$$

$$df_t = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\left(\frac{\partial \Psi_1}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi_1}{\partial w^2}\right) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\left(\frac{\partial \Psi_1}{\partial w}\right)^2\right) dt + \frac{\partial f}{\partial w}\frac{\partial \Psi_1}{\partial w} dW_t \tag{2.8}$$

The self-financing condition is

$$S_t df_t + B_t dc_t = \frac{1}{2}\left(\frac{\partial f}{\partial S}(S_t, t)\frac{\partial \Psi_1}{\partial w}(W_t, t)\right)^2 \frac{\partial \Psi}{\partial f}(f_t, W_t, t) dt \tag{2.9}$$

Proof. Given that $S_t = \Psi_1(W_t, t)$ and $f_t = f(\Psi_1(W_t, t), t)$ conditions (2.7) and (2.8) are obtained using the Ito's rule. Consider that the transactions are realized at the sequence of instants $t_1 < t_2 < t_3 < \ldots < t_n$ such that $t_i \in [0, T]$ and $t_i = \frac{i}{n}T$ for $i = 1, 2, \ldots, n$. The function f_t can be approximated by $\sum_{i=1}^{n} f_{i-1}\chi_{[t_{i-1}, t_i]}(t)$ with $f_{i-1} = f(S_{t_{i-1}}, t_{i-1})$. Moreover, we use the following notation $f_i := f_{t_i}$ and $f_{i-1} := f_{t_i^-}$. Using condition (2.4) for each transaction and applying it the Taylor expansion at the function $\Psi(f, w, t)$ we obtain

$$\begin{aligned}
(c_t - c_{t-})B_t &= -\int_{f_{i-1}}^{f_i} \Psi(x, W_t, t) dx \\
&= -\int_{f_{i-1}}^{f_i} \Psi(f_i, W_{t_i}, t_i) - \frac{\partial \Psi}{\partial f}(f_i, W_{t_i}, t_i)(x - f_i) + o(|f_i - f_{i-1}|) dx \\
&= -S_{t_i} df_{t_i} + \frac{1}{2}\frac{\partial \Psi}{\partial f}(f_i, W_{t_i}, t_i)(df_{t_i})^2 + o(|f_i - f_{i-1}|^2).
\end{aligned}$$

Using Itô's Lemma and letting $|t_i - t_{i-1}| \to 0$, the result follows. □

3. Collective behaviour. In general the models of illiquid markets have theoretical difficulties, because they predict the collapse of the market at least in some situations. The existence of large traders with capacity to influence the market price or the existence of traders with privileged information allows market manipulation. We say that there is a market manipulation when there is some trading strategy that allows to move the price to make risk free profit. If such strategies can be used without limitations, the market collapses because small traders are stripped of their wealth. We say that a model is consistent if there are neither arbitrage possibilities nor possibilities of market manipulation. There are some works as [3] and [4] studying market manipulation. Also in [6] these questions are discussed and it is shown that the Wilomott-Schönbucher model is not, in general, consistent. Despite the consistency problems mentioned above, the Wilmott-Schönbucher model is attractive to study the consequences of collective behaviour, for a particular case discussed in [6]. We deduce the dynamic of a self-financed according to a proposal presented in last section. Finally, we derive the Black-Scholes equation for the true value of an option in the presence of collective behaviour by a large group of traders in the market.

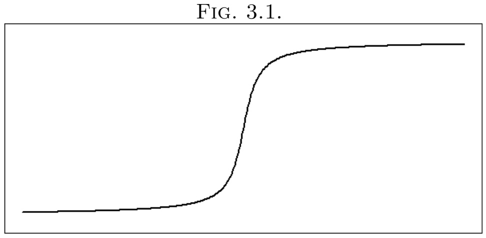
3.1. Synchronized hedgers. Consider a market with a large number of small traders and no large trader. Suppose that a sizeable fraction of these small traders are following similar strategies. Since all traders in this group sell and buy in similar circumstances, they act collectively like a large trader without being aware of this fact. Situations of this kind may arise in real markets due to the widespread use of model-assisted and automatic trading when large numbers of traders use similar models or algorithms.

In this case, the issues related to market manipulation do not apply because the individual traders are unaware of their mutual synchronization and are competitors. The Wilmott-Schönbucher model is attractive to study the consequences of such collective behaviour, due to its relative simplicity and "first principles" approach.

Here we assume that the excess of demand function is of the form $\chi(S_t, W_t, t) = \alpha(S_t^* - S_t)$, where $\alpha > 0$ is a constant and S_t^* verifies $dS_t^* = \theta S_t^* dt + \nu S_t^* dW_t$. We assume also that the synchronized small traders are hedgers, who try to replicate an European put option with strike price K, using the Black-Scholes strategy. So, this group of hedgers acts like a large trader with delta strategy $f(S_t, t) = N(d_1) - 1$, where $N(d_1) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{d_1} \exp\left(-\frac{z^2}{2}\right)$ and $d_1 = \frac{\log(\frac{S_t}{K}) + (r + \frac{\nu^2}{2})(T-t)}{\nu\sqrt{T-t}}$.

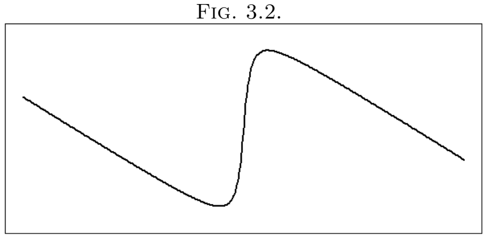
3.2. The price process. It is important to understand the price mechanism in our example to deduce the Black-Scholes equation for this market. The geometric shape of the strategy of the large trader varies as the time approaches to maturity. Near the maturity the strategy of the large trader approaches a step function with step from -1 to 0 as shown in Figure 3.1.

FIG. 3.1.



On the other hand, the function excess of demand is linear. So, near the maturity, the function $S \mapsto \chi(S, W, t) + f(S, t)$ is S-shaped as shown in Figure 3.2.

FIG. 3.2.



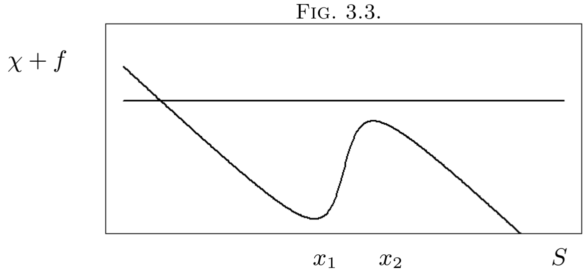
This function has one local minimizer and one local maximizer which we denote by $x_1(t)$ and $x_2(t)$, respectively. We also define the points $\Sigma_1(t)$, $\Sigma_2(t)$ as the unique solutions to

$$\begin{cases} \chi(S,W,t)+f(S,t)=\chi(x_i(t),W,t)+f(x_i(t),t) \\ S\neq x_i(t) \end{cases} \tag{3.1}$$

$i=1,2$. Notice that in our case $x_1(t),x_2(t),\Sigma_1(t),\Sigma_2(t)$ do not depend on W. Now we discuss the price equilibria.

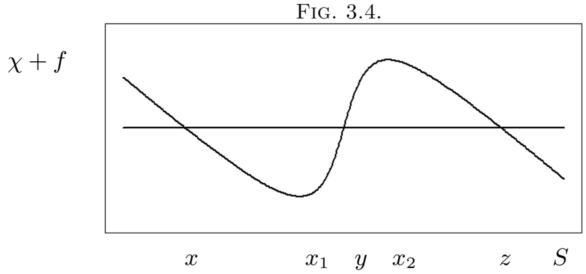
The sequence of Figures 3.3,3.4 and 3.5 illustrates one possible sequence for the price equilibrium. In Figure 3.3, there is a unique equilibrium price, $x(t)$ and any small perturbation caused by the Brownian motion moves only a little the equilibrium price.

Fig. 3.3.



In the Figure 3.4 there are three possibles equilibria, $x(t),y(t)$ and $z(t)$. The middle equilibrium, $y(t)$, is unstable. Notice that around $y(t)$ the slope of $\chi(S,W,t)+f(S,t)$ is positive. This means that the bigger the positive price variation is, the greater the positive excess of demand variation gets. So, at price $y(t)$ any perturbation moves the equilibrium price to $x(t)$, or to $z(t)$, which are stable equilibria.

Fig. 3.4.



Finally consider the case when there are two equilibria (Figures 3.5 and 3.6). In figure 3.5 the equilibria are $x_1(t)$ and $\Sigma_1(t)$. $\Sigma_1(t)$, is stable and acts as the equilibrium price in the Figure 3.3. The equilibrium, $x_1(t)$, is stable with respect to negative price perturbation but any positive price perturbation moves the equilibrium price to a price in the positive slope and consequently to $\Sigma_1(t)$. So we consider that in the first moment t, where the market price reaches $x_1(t)$ from the left, there is a jump in the price process from $S_{t-}=x_1(t)$ to $S_t=\Sigma_1(t)$.

The case in figure 3.6 is analogous: in the first moment the price reaches $x_2(t)$ from the right there is a price jump from $S_{t-}=x_2(t)$ to $S_t=\Sigma_2(t)$.

Fig. 3.5.

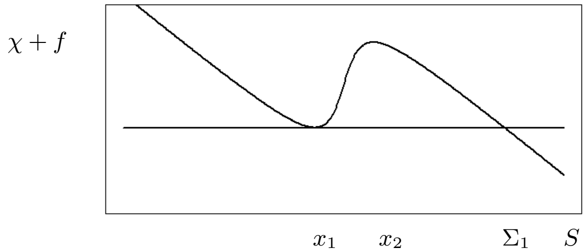
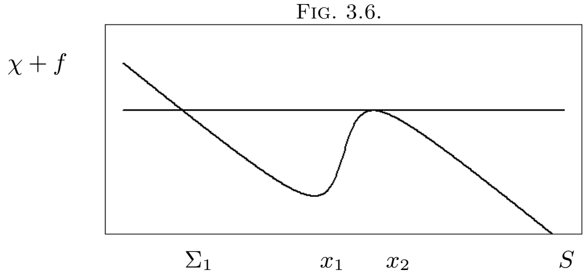


Fig. 3.6.



3.3. Portfolio dynamic. In this subsection we would like to obtain the dynamic of a portfolio that is self-financing in the in the collective behaviour model. To simplify the calculation we set $B_t \equiv 1$. Given the particular characteristics of the price process for the collective behaviour model, we need to define the stochastic process carefully. Let $\Gamma_i(W,t)$, with $i = 1, 2$, be the functions:

$$\Gamma_1(W,t) = \min\{S : \alpha(S^* - S) + f(S,t) = 0\} \tag{3.2}$$

$$\Gamma_2(W,t) = \max\{S : \alpha(S^* - S) + f(S,t) = 0\} \tag{3.3}$$

So, we can define the cadlag process (S_t, I_t), where:

$$I_t = \begin{cases} 1 & \text{if} \quad S_{t-} < x_1(t) \vee S_{t-} = x_2(t) \\ 2 & \text{if} \quad S_{t-} > x_2(t) \vee S_{t-} = x_1(t) \end{cases} \tag{3.4}$$

$$S_t = \Gamma_{I_t}(W_t, t). \tag{3.5}$$

The strategy of the big portion of the small investors is $\phi(W_t,t) = f\left(\Gamma_{I_t}(W_t,t),t\right)$. In consequence, the value of the portfolio is

$$Y_t = \phi(W_t,t)\Gamma_{I_t}(W_t,t) + c_t, \tag{3.6}$$

and the dynamics of Y_t is obtained by the Itô Lemma:

$$\begin{aligned} dY_t =& d\left(\phi(W_t,t)\Gamma_{I_t}(W_t,t)\right) + dc_t \\ =& \Gamma_{I_t}(W_t,t)d\phi(W_t,t) + \phi(W_t,t)d\Gamma_{I_t}(W_t,t) + d\phi(W_t,t)d\Gamma_{I_t}(W_t,t) + dc_t \end{aligned} \tag{3.7}$$

There are 4 possibles scenarios:

$$I_{t-} = 1; I_t = 1 \longrightarrow S_{t-} < x_1(t) \tag{3.8}$$

$$I_{t-} = 2; I_t = 1 \longrightarrow S_{t-} = x_2(t) \tag{3.9}$$

$$I_{t-} = 1; I_t = 2 \longrightarrow S_{t-} = x_1(t) \tag{3.10}$$

$$I_{t-} = 2; I_t = 2 \longrightarrow S_{t-} > x_2(t) \tag{3.11}$$

To guarantee that our self-financing condition for illiquid markets is verified we impose the condition (2.9) for our model, and we obtain

$$dc_t = -\left(\Gamma_{I_t}(W_t,t) - \frac{1}{2\alpha} d\phi_t\right) d\phi_t$$

To calculate the dynamics of $\Gamma_{I_t}(W_t,t)$ we notice that:

$$\begin{aligned} d\Gamma_{I_t} &= \Gamma_{I_t}(W_t,t) - \Gamma_{I_{t-}}(W_{t-},t-) \\ &= \Gamma_{I_t}(W_t,t) - \Gamma_{I_{t-}}(W_t,t) + \Gamma_{I_{t-}}(W_t,t) - \Gamma_{I_{t-}}(W_{t-},t-) \\ &= \Gamma_{I_t}(W_t,t) - \Gamma_{I_{t-}}(W_t,t) + \left(\frac{\partial \Gamma_{I_{t-}}}{\partial t} + \frac{\partial^2 \Gamma_{I_{t-}}}{\partial W^2}\right) dt + \frac{\partial \Gamma_{I_{t-}}}{\partial W} dW_t \end{aligned}$$

where the difference of two first terms can be simplified in:

$$\Gamma_{I_t}(W_t,t) - \Gamma_{I_{t-}}(W_t,t) = \begin{cases} 0 & \text{if } \ I_{t-} = I_t \\ \Gamma_{I_t}(W_t,t) - x_2(t) & \text{if } \ I_{t-} = 2 \ \text{ and } \ I_t = 1 \\ \Gamma_{I_t}(W_t,t) - x_1(t) & \text{if } \ I_{t-} = 1 \ \text{ and } \ I_t = 2. \end{cases}$$

To derive the dynamics of ϕ_t we need to observe the 4 different scenarios:

$$d\phi_t = \phi_t - \phi_{t-} = \left\{ \begin{array}{ll} f\left(\Gamma_1(W_t,t),t\right) - f\left(\Gamma_1(W_{t-},t-),t-\right) & \text{if} \quad S_{t-} < x_1(t) \\ f\left(\Gamma_1(W_t,t),t\right) - f\left(\Gamma_2(W_{t-},t-),t-\right) & \text{if} \quad S_{t-} = x_2(t) \\ f\left(\Gamma_2(W_t,t),t\right) - f\left(\Gamma_1(W_{t-},t-),t-\right) & \text{if} \quad S_{t-} = x_1(t) \\ f\left(\Gamma_2(W_t,t),t\right) - f\left(\Gamma_2(W_{t-},t-),t-\right) & \text{if} \quad S_{t-} > x_2(t) \end{array} \right. .$$

In the scenario (3.8) and (3.11) we have that:

$$\begin{aligned} f\left(\Gamma_i(W_t,t),t\right) - f\left(\Gamma_i(W_{t-},t-),t-\right) &= \frac{\partial f}{\partial \Gamma_i} d\Gamma_i + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial \Gamma_i^2}\left(d\Gamma_i\right)^2 \\ &= \frac{\partial f}{\partial \Gamma_i}\left(\frac{\partial \Gamma_i}{\partial W} dW_t + \frac{\partial \Gamma_i}{\partial t} dt + \frac{1}{2}\frac{\partial^2 \Gamma_i}{\partial W^2} dt\right) + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial \Gamma_i^2}\left(\frac{\partial \Gamma_i}{\partial W}\right)^2\right) dt \\ &= \left(\frac{\partial f}{\partial \Gamma_i}\frac{\partial \Gamma_i}{\partial t} + \frac{1}{2}\left(\frac{\partial f}{\partial \Gamma_i}\frac{\partial^2 \Gamma_i}{\partial W^2} + \frac{\partial^2 f}{\partial \Gamma_i^2}\left(\frac{\partial \Gamma_i}{\partial W}\right)^2\right) + \frac{\partial f}{\partial t}\right) dt + \frac{\partial f}{\partial \Gamma_i}\frac{\partial \Gamma_i}{\partial W} dW_t \\ &= \mu_i(t-,S_{t-})dt + \sigma_i(t-,S_{t-})dW_t \end{aligned}$$

When $S_{t-} = x_2(t)$ the dynamics of the self-financing strategy is:

$$\begin{aligned} f\left(\Gamma_1(W_t,t),t\right) - f\left(\Gamma_2(W_{t-},t-),t-\right) &= f\left(\Gamma_1(W_t,t),t\right) - f\left(\Gamma_2(W_t,t),t\right) \\ &\quad + f\left(\Gamma_2(W_t,t),t\right) - f\left(\Gamma_2(W_{t-},t-),t-\right) \\ &= f\left(\Gamma_1(W_t,t),t\right) - f\left(x_2(t),t-\right) + \mu_2(t-,S_{t-})dt + \sigma_2(t-,S_{t-})dW_t \end{aligned}$$

Finally we have the situation $S_{t-} = x_1(t)$ and the dynamics is:

$$\begin{aligned} f\left(\Gamma_2(W_t,t),t\right) - f\left(\Gamma_1(W_{t-},t-),t-\right) &= f\left(\Gamma_2(W_t,t),t\right) - f\left(\Gamma_1(W_t,t),t\right) \\ &+ f\left(\Gamma_1(W_t,t),t\right) - f\left(\Gamma_1(W_{t-},t-),t-\right) \\ &= f\left(\Gamma_2(W_t,t),t\right) - f\left(x_1(t),t-\right) + \mu_1(t-,S_{t-})dt + \sigma_1(t-,S_{t-})dW_t \end{aligned}$$

Notice that $\frac{\partial \Gamma_i}{\partial W}$ is unbounded. We have the equilibrium condition (2.3) from which we can write $S(W,t) = g(W,t)$. So, the implicit function theorem guarantees that:

$$\frac{\partial \Gamma_i}{\partial W} = -\frac{\partial\left(\chi(S,W,t) + f(S,t)\right)}{\partial W}\left(\frac{\partial\left(\chi(S,W,t) + f(S,t)\right)}{\partial S}\right)^{-1}.$$

When $S \to x_1(t)$ or $S \to x_2(t)$ the denominator tends to 0, then $\frac{\partial \Gamma_i}{\partial W}$ is unbounded. For each scenario we can specify a little more the dynamics of c_t. When we have $S_{t-} < x_1(t)$ or $S_{t-} > x_2(t)$ the dynamics of c_t is:

$$\begin{aligned} dc_t &= -\left(\Gamma_{I_t}(W_t,t) - \frac{1}{2\alpha}d\phi_t\right)d\phi_t \\ &= -\Gamma_{I_t}(W_t,t)d\phi_t + \frac{1}{2\alpha}\sigma_i^2(t-,S_{t-})dt \end{aligned}$$

for $i = 1$ or $i = 2$ respectively. There are two other cases $S_{t-} = x_1(t)$ and $S_{t-} = x_2(t)$. We derive the dynamics of c_t for the $S_{t-} = x_1(t)$:

$$\begin{aligned} dc_t &= -\left(\Gamma_{I_t}(W_t,t) - \frac{1}{2\alpha}d\phi_t\right)d\phi_t \\ &= -\left(\Gamma_{I_t}(W_t,t) - \frac{1}{2\alpha}\left(f(\Gamma_2(W_t,t),t)) - f(x_1(t),t) + \mu_1(t-,S_{t-})dt + \sigma_1(t-,S_{t-})dW_t\right)\right) \\ &\times\left(f(\Gamma_2(W_t,t),t)) - f(x_1(t),t) + \mu_1(t-,S_{t-})dt + \sigma_1(t-,S_{t-})dW_t\right) \\ &= -\Gamma_{I_t}(W_t,t)d\phi_t - \frac{1}{2\alpha}\sigma_1(t-,S_{t-})\left(f(\Gamma_2(W_t,t),t)) - f(x_1(t),t)\right)dW_t \\ &- \frac{1}{2\alpha}\left(\mu_1(t-,S_{t-})\left(f(\Gamma_2(W_t,t),t)) - f(x_1(t),t)\right) + \sigma_1^2(t-,S_{t-})\right)dt - \frac{1}{2\alpha}\left(f(\Gamma_2(W_t,t),t)) - f(x_1(t),t)\right)^2 \end{aligned}$$

for the case $S_{t-} = x_2(t)$ the derivation of the dynamics of c_t is similar.

3.4. The Black-Scholes equation. In this section we want to derive the the Black-Scholes equation for the collective behaviour in the Wilmott-Schönbucher model.

First we present the dynamic of the price process:

$$\begin{aligned} \sigma(S_t,t) &= \frac{\alpha\nu S^*}{\alpha - \frac{\partial}{\partial S}f(S_t,t)} \\ \mu(S_t.t) &= \frac{1}{\alpha - \frac{\partial}{\partial S}f(S_t,t)}\left(\alpha\theta S^* + \frac{\partial}{\partial t}f(S_t,t) + \frac{1}{2}\sigma^2(S_t,t)\frac{\partial^2}{\partial S^2}f(S_t,t)\right) \end{aligned}$$

We notice that the authors write the drift of the diffusion as a function of S_t and t. Indeed, we can write the Brownian Motion, W_t, as a function of S_t and t. So, it's easy to show that the drift and the volatility of the diffusion is given by:

$$\begin{aligned} \sigma(S_t,t) &= \frac{\alpha\nu\left(S - f(S_t,t)\right)}{\alpha - \frac{\partial}{\partial S}f(S_t,t)} \\ \mu(S_t.t) &= \frac{1}{\alpha - \frac{\partial}{\partial S}f(S_t,t)}\left(\alpha\theta\left(S - f(S_t,t)\right) + \frac{\partial}{\partial t}f(S_t,t) + \frac{1}{2}\sigma^2(S_t,t)\frac{\partial^2}{\partial S^2}f(S_t,t)\right) \end{aligned}$$

If we consider that there are no jumps between t^- and t then, we can obtain the dynamics of the self-financing strategy value such as in the last section. This dynamics can be simplified by considering

$$dY_t = a(S_t,t)dt + b(S_t,t)dW_t \tag{3.12}$$

Then, the HJB equation and the usually boundary condition are is given by

$$V_t(t,S,Y) + \mathcal{L}V(t,S,Y) = 0, \quad \forall (t,S,Y) \in [0,T[\times \mathbb{R}^+ \times \mathbb{R} \tag{3.13}$$

$$V(T,S) = \Phi(S,Y), \quad \forall (S,Y) \in \mathbb{R}^+ \times \mathbb{R} \tag{3.14}$$

where $\mathcal{L}$ is the infinitesimal generator, i.e., the operator defined as:

$$\begin{aligned}\tilde{\mathcal{L}}g =& \lim_{h\to 0^+} \frac{E_t^{s,y}\left[g(S_{t+h},Y_{t+h})\right] - g(s,y)}{h}\\ =& \frac{\partial g}{\partial y}(S_t,Y_t)a(S_t,t) + \frac{\partial g}{\partial s}(S_t,Y_t)\mu(S_t,t) + \frac{\partial^2 g}{\partial y \partial s}(S_t,Y_t)\sigma(S_t,t)b(S,t)\\ +& \frac{1}{2}\left(\frac{\partial^2 g}{\partial y^2}(Y_t,S_t)b^2(S_t,t) + \frac{\partial^2 g}{\partial s^2}(Y_t,S_t)\sigma^2(S_t,t)\right)\end{aligned} \tag{3.15}$$

for g smooth and bounded.

In the scenario (3.10) we have a jump when the price reaches the minimum price $x_1(t)$. So, in the moment t, the value of S attains the value $x_1(t)$ and jumps to the value $\Sigma_1(t)$. We need to compute the jump in the self-financing strategy value:

$$\begin{aligned}Y_t =& \; Y_{t-} - \left(x_1(t) + \frac{f(\Sigma_1(t),t) - f(x_1(t),t)}{2\alpha}\right)\big(f(\Sigma_1(t),t) - f(x_1(t),t)\big)\\ +& \; f(\Sigma_1(t),t)\Sigma_1(t) - f(x_1(t),t)x_1(t)\\ =& \; Y_{t-} - \alpha\left(x_1(t) + \frac{\Sigma_1(t) - x_1(t)}{2}\right)\big(\Sigma_1(t) - x_1(t)\big)\\ +& \; f(\Sigma_1(t),t)\Sigma_1(t) - f(x_1(t),t)x_1(t)\\ =& Y_{t-} - \alpha\frac{\Sigma_1(t) + x_1(t)}{2}\big(\Sigma_1(t) - x_1(t)\big) + f(\Sigma_1(t),t)\Sigma_1(t) - f(x_1(t),t)x_1(t)\end{aligned} \tag{3.16}$$

On the other hand, in the scenario (3.11), in the moment t the price process S_t attains $x_2(t)$ and jumps to the value $\Sigma_2(t)$.

The jump in the self-financing strategy value is given by:

$$Y_t = Y_{t-} - \alpha\frac{\Sigma_2(t) + x_2(t)}{2}\big(\Sigma_2(t) - x_2(t)\big) + f(\Sigma_2(t),t)\Sigma_2(t) - f(x_2(t),t)x_2(t) \tag{3.17}$$

Then the Black-Scholes equation is (3.13) and (3.14) adding the conditions:

$$\begin{aligned}V(t,x_1(t),y) =& \; V\left(t,\Sigma_1(t), y - \alpha\frac{\Sigma_1(t) + x_1(t)}{2}\big(\Sigma_1(t) - x_1(t)\big)\right.\\ & \left. + f(\Sigma_1(t),t)\Sigma_1(t) - f(x_1(t),t)x_1(t)\right), \; \forall (t,Y) \in [0,T[\times\mathbb{R}\end{aligned} \tag{3.18}$$

$$\begin{aligned}V(t,x_2(t),y) =& \; V\left(t,\Sigma_2(t), y - \alpha\frac{\Sigma_2(t) + x_2(t)}{2}\big(\Sigma_2(t) - x_2(t)\big)\right.\\ & \left. + f(\Sigma_2(t),t)\Sigma_2(t) - f(x_2(t),t)x_2(t)\right), \; \forall (t,Y) \in [0,T[\times\mathbb{R}\end{aligned} \tag{3.19}$$

The jumps in the price process suggest that the solution, if it exists, is not continuous. Observe the Figure 3.6 where we try to illustrate the boundary conditions (3.14), (3.18) and (3.19). Suppose that there is a continuous solution, then $\lim_{S\to x_1(t)^-}\lim_{t\to T^-} V(t,S,Y) = \lim_{t\to T^-}\lim_{S\to x_1(t)^-} V(t,S,Y)$. However this is not verified. Notice that,

$$\lim_{S\to x_1(t)^-}\lim_{t\to T^-} V(t,S,Y) = V(T,K,Y)$$
$$\lim_{t\to T^-}\lim_{S\to x_1(t)^-} V(t,S,Y) = V(T,K+\frac{1}{\alpha},Y-\frac{1}{2\alpha})$$

where K is the strike price. Naturally, $V(T,K,Y) = V(T,K+\frac{1}{\alpha},Y-\frac{1}{2\alpha})$ for all $Y\in\mathbb{R}$ is not verified.

Appendix: Calculation of the maximum and minimum points in collective behaviour case. When the variables W and t are fixed we can find the extreme points using the usual tools for one variable function. We start with the calculation of critical points.

$$\begin{aligned}\frac{\partial}{\partial S}\left\{\chi(S_t,W_t,t)+f(S_t,t)\right\} &= 0\\ -\alpha+\frac{1}{\sqrt{2\Pi}}\exp\left(-\frac{d_1^2}{2}\right)\frac{\partial}{\partial S}d_1 &= 0\\ \frac{1}{S_t\nu\sqrt{2\Pi(T-t)}}\exp\left(-\frac{d_1^2}{2}\right) &= \alpha.\end{aligned}$$

In order to simplify we set $A(t)=\nu\sqrt{2\Pi(T-t)}$. Therefore,

$$-\frac{d_1^2}{2} = \log(\alpha A(t))+\log(S_t) \tag{3.20}$$

$$-\frac{1}{2}\left(\frac{\log(S_t)}{B(t)}+C(t)\right)^2 = \log(\alpha A(t))+\log(S_t), \tag{3.21}$$

where $B(t)=\nu\sqrt{T-t}$ and $C(t)=\frac{(r+\frac{\nu^2}{2})(T-t)-\log(K)}{B(t)}$. If we use the substitution $\log(S_t)=y$, we will have a second order equation:

$$-\frac{1}{2B^2(t)}y^2-(1+\frac{C(t)}{B(t)})y-(\frac{C^2(t)}{2}+\log(\alpha A))=0$$

So, we have $y=\frac{-\left(1+\frac{C(t)}{B(t)}\right)\pm\sqrt{\left(1+\frac{C(t)}{B(t)}\right)^2-\frac{C^2(t)}{B^2(t)}-\frac{2\log(\alpha A)}{B^2(t)}}}{\frac{1}{B^2(t)}}$. After some simplifications we can obtain $y=-E(t)\pm\sqrt{B^4(t)D(t)-2B^2(t)\log(\alpha A)}$. Here $E(t)=(r+\frac{3}{2}\nu^2)(T-t)-\log(K)$ and $D(t)=\frac{2}{\nu^2}(\frac{\log(K)}{T-t}+r+\nu^2)$. Then the minimum and maximum points are given by,

$$S_t=\exp\left(-E(t)\pm\sqrt{B^4(t)D(t)-2B^2(t)\log(\alpha A)}\right). \tag{3.22}$$

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